

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

# **An Examination of Liquidity Risk and Liquidity Risk Measures**

by

**Aneez Bhyat**

Supervised by Prof Ronald Becker

A dissertation submitted in partial fulfilment of the requirements for the

Master of Science in Financial Mathematics  
Department of Mathematics and Applied Mathematics  
Faculty of Science  
University of Cape Town, Cape Town  
July 2010

### **Plagiarism Declaration**

I know the meaning of Plagiarism and declare that all of the work in this document, save for that which is properly acknowledged, is my own.

University of Cape Town

## **Abstract**

Liquidity risk represents a vacuum of rigour in the otherwise well-researched area of risk management. In both practice and theory most of finance is silent regarding its scope and effect. This is principally due to a lack of consensus regarding its definition and measurement. Current liquidity risk measures differ fairly widely in both respects. This thesis attempts at addressing this by consolidating and examining the principle liquidity risk measures used in financial literature.

The goal is threefold: distil a clear definition for liquidity risk, mould organic groupings between the measures based on similarities of purpose and assess them in terms of both accuracy and practicality.

The result of the investigation shows that liquidity risk is composed of an array of inter-related aspects, all of which are important to its effect. Liquidity has endemic effects which must be managed. While current approaches to its management exist, none of them capture the full extent of liquidity risk. Standard liquidity measures only capture a specific aspect of the risk and while they do seem to provide information on market-wide liquidity, they lack predictive power. Contrastingly while the more complex Liquidity-Value-at-Risk models aim at modelling the totality of liquidity risk, with the exception of one model, their adjustments do not make them more accurate than standard Value-at-Risk.

## Acknowledgements

I would like to thank Zishan, my partner and friend for never letting me give up.

I would like to thank Prof Becker for his unwavering tutelage, encouragement and guidance over the course of the Masters programme.

I would like to thank my family for leading me here.

University of Cape Town

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Towards a Definition of Liquidity Risk</b>	<b>5</b>
2.1	The Price Formation Process . . . . .	7
2.2	Intuitive definitions . . . . .	8
2.3	Transaction Cost-Based Liquidity Risk . . . . .	10
2.3.1	The Bid-Ask Spread . . . . .	10
2.3.2	Price Risk . . . . .	13
2.3.3	Opportunity Costs . . . . .	14
2.3.4	Price-Impact Costs . . . . .	15
2.4	Other Aspects of Liquidity Risk . . . . .	21
2.5	Liquidity under Stress . . . . .	24
2.6	Conclusive Definition . . . . .	26
<b>3</b>	<b>The Importance of Liquidity-Risk Management</b>	<b>30</b>
3.1	The Magnitude of Transaction Costs . . . . .	31
3.2	Liquidity, Solvency and Systematic Crises . . . . .	32

3.3	Liquidity Crises Case Studies LTCM and Amaranth Advisors	35
3.3.1	Long-Term Capital Management	35
3.3.2	Amaranth Advisors LLC	37
3.4	Model Accuracy	38
3.5	Problems with Value-at-Risk	41
3.5.1	Parametric Modelling	43
3.5.2	Historical Simulation	45
3.5.3	Kernel Estimators	46
3.5.4	Monte Carlo Estimation	48
3.5.5	Extreme-Value Theory Adjustments	48
3.5.6	Comparing the Methodologies	51
3.5.7	VaR and Liquidity-Related Risk	53
3.6	The Elements of a Good Liquidity Measure and Model	54
<b>4</b>	<b>The Gamut of Liquidity Risk Measures and Models</b>	<b>58</b>
4.1	Spread-Based Measures	59
4.2	Trading-Activity-Based Measures	61
4.3	Price-Impact Measures	64
4.4	Assessing the Measures	66
4.5	Liquidity-VaR Models	72
4.5.1	Spread-Adjustment Models	73
4.5.2	Parametric-Adjustment Models	87

4.5.3	Optimal Liquidation Strategy Models . . . . .	95
4.6	Model Comparisons . . . . .	111
<b>5</b>	<b>Model Implementation and Analysis</b>	<b>120</b>
5.1	Data & Methodology . . . . .	120
5.1.1	The JSE . . . . .	122
5.1.2	Data . . . . .	124
5.1.3	Methodology . . . . .	126
5.2	Results . . . . .	136
5.2.1	Measures Analysis . . . . .	136
5.2.2	L-VaR Analysis . . . . .	154
5.2.3	Synopsis . . . . .	192
<b>6</b>	<b>Conclusion</b>	<b>195</b>
<b>A</b>	<b>The Average-Weighted Spread</b>	<b>206</b>
<b>B</b>	<b>GMM Estimation</b>	<b>207</b>
<b>C</b>	<b>Statistical Backtesting of Value-at-Risk</b>	<b>210</b>
<b>D</b>	<b>VBA Code</b>	<b>214</b>



# List of Figures

2.1	Price Risk [65]. . . . .	13
2.2	Price Risk over Time [65]. . . . .	14
2.3	Opportunity Cost Trade-Off [69]. . . . .	16
2.4	Price Impacts [69]. . . . .	17
2.5	Pre-Trade Price Impacts [65]. . . . .	17
2.6	Temporary vs Permanent Price Impacts [49]. . . . .	20
2.7	Aspects of Market Liquidity [74]. . . . .	22
3.1	Price Impact Costs [65]. . . . .	31
3.2	Smoothing with Kernel Estimators [52]. . . . .	47
4.1	Time Series plots of Volume [43]. . . . .	67
4.2	Spread Distributions [14]. . . . .	75
4.3	Parametric L-VaR [14]. . . . .	76
4.4	Alternate Price Impact Formulations [65] . . . . .	109
5.1	Trading Strategy Algorithm . . . . .	130
5.2	FTSE/JSE All Share Index . . . . .	138

5.3	AGL Liquidity Measures - Trading Activity Variables . . . .	139
5.4	AGL Liquidity Measures - B/A Spread & Scaled Volume . .	140
5.5	AGL Liquidity Measures - Price Impact Variables . . . . .	141
5.6	PIK Liquidity Measures - Trading Activity Variables . . . .	142
5.7	PIK Liquidity Measures - B/A Spread & Scaled Volume . . .	143
5.8	PIK Liquidity Measures - Price Impact Variables . . . . .	144
5.9	PIK Liquidity Measures - Price Impact Variables . . . . .	145
5.10	CML Liquidity Measures - Trading Activity Variables . . . .	146
5.11	CML Liquidity Measures - B/A Spread & Scaled Volume . .	147
5.12	CML Liquidity Measures - Price Impact Variables . . . . .	148
5.13	SKY Liquidity Measures - Trading Activity Variables . . . .	149
5.14	SKY Liquidity Measures - B/A Spread & Scaled Volume . .	151
5.15	SKY Liquidity Measures - Price Impact Variables . . . . .	152
5.16	AGL Standard VaR Models . . . . .	156
5.17	PIK Standard VaR Models . . . . .	158
5.18	CML Standard VaR Models . . . . .	159
5.19	SKY Standard VaR Models . . . . .	160
5.20	AGL Spread-Adjustment VaR Models . . . . .	161
5.21	PIK Spread-Adjustment VaR Models . . . . .	162
5.22	CML Spread-Adjustment VaR Models . . . . .	163
5.23	SKY Spread-Adjustment VaR Models . . . . .	164
5.24	CML Spread and VaR models . . . . .	165

5.25	SKY Spread and VaR Models . . . . .	166
5.26	AGL Parametric-Adjustment VaR Models . . . . .	167
5.27	PIK Parametric-Adjustment VaR Models . . . . .	168
5.28	CML Parametric-Adjustment VaR Models . . . . .	169
5.29	SKY Parametric-Adjustment VaR Models . . . . .	170
5.30	AGL Optimal Liquidation Strategy VaR Models . . . . .	171
5.31	PIK Optimal Liquidation Strategy VaR Models . . . . .	172
5.32	CML Optimal Liquidation Strategy VaR Models . . . . .	173
5.33	SKY Optimal Liquidation Strategy VaR Models . . . . .	174
5.34	Average & Standard Deviation of Price Impact Parameters .	175
5.35	AGL VaR Model Comparison . . . . .	176
5.36	PIK VaR Model Comparison . . . . .	177
5.37	CML VaR Model Comparison . . . . .	178
5.38	SKY VaR Model Comparison . . . . .	179
5.39	Average VaR Forecasts across Models . . . . .	180
5.40	Average VaR Forecasts across Models . . . . .	181
5.41	Variance of VaR Forecasts across Models . . . . .	182
5.42	Percentage Profit & Loss Coverage by VaR Models . . . . .	188
A.1	Weighted-Average Spread[60]. . . . .	206

# List of Tables

2.1	Depth vs Breadth . . . . .	21
3.1	Duffie Test Results . . . . .	40
4.1	Price-Impact Functions [6] . . . . .	103
4.2	Model Data Requirements 1 . . . . .	118
4.3	Model Data Requirements 2 . . . . .	119
5.1	Brokerage Costs . . . . .	131
5.2	Major Market Crises . . . . .	137
5.3	AGL - Descriptive Statistics . . . . .	145
5.4	PIK - Descriptive Statistics . . . . .	146
5.5	CML - Descriptive Statistics . . . . .	147
5.6	SKY - Descriptive Statistics . . . . .	148
5.7	Dickey-Fuller Unit Root Test Results . . . . .	149
5.8	Dickey-Fuller Unit Root Test Results - Level vs Differenced Series . . . . .	149
5.9	Correlation Analysis Results . . . . .	150

5.10 Factor Analysis Results . . . . .	153
5.11 Summary Micro-Structure Statistics . . . . .	155
5.12 Sensitivity Analysis 1 . . . . .	183
5.13 Sensitivity Analysis 2 . . . . .	184
5.14 Statistical Backtesting Results 1 . . . . .	190
5.15 Statistical Backtesting Results 2 . . . . .	190
5.16 Statistical Backtesting Results 3 . . . . .	191

University of Cape Town

# Chapter 1

## Introduction

*“Portfolios are marked-to-market at the middle of the bid-offer spread and many hedge funds use models that incorporate this assumption. In late August, there was only one realistic value for the portfolio: the bid price. Amid such major sell-offs only the first seller obtains a reasonable price for its security, the rest lose a fortune having to pay a liquidity premium” – Meriwether’s Meltdown [14]*

Liquidity risk loosely refers to the risk that an investor faces in not realizing the expected proceeds upon liquidation. It differs from standard market-related risk of loss in that it is meant to refer solely to losses borne off liquidation.

Generally these losses stem from an array of factors including, aggregate market conditions, investor preferences, macro-economic events and, even the manner in which an investor trades. The risk is multi-faceted and driven by the complex factors that define the flow of information in a market and determine how prices are set. It is inherent to every trade executed and crucially determines the optimal stability and efficiency of markets and the financial system as a whole.

Liquidity is pivotal to the normal functioning of a financial system. Its growing importance, particularly in the increasingly integrated global market, has been highlighted by the series of high-profile firm failures and economic distresses which have been initiated by liquidity risk mismanagement and model failure. The 1987 stock market crash, the Dec 1997 run on the Thai Baht, the Oct 1998 Long-Term Capital Management bankruptcy<sup>1</sup> and Rus-

---

<sup>1</sup>whose director, John Meriwether, is referred to in the opening quote

sian default and the more recent recession-generating financial collapse, all have as their root a liquidity crises with endemic effects.

Despite the grave importance of these events for the world economy, none of them have spurred on the release of regulation to prevent their recurrence. Moreover, more specific research into liquidity-risk management techniques has only recently been undertaken as studies in risk have preferred to focus on Value-at-Risk. Prudence has in some sense been undermined by severe model dependency and the pretence that all risks are captured in Value-at-Risk and other ad-hoc risk measures.

In effect liquidity-risk has largely been forgotten in the pursuit of partially forecasting market-related risk. This is possibly due to the confusion surrounding what precisely constitutes the risk and the fact that proper and robust mathematical treatments of it, although they exist, have not yet gained traction due to uncertainty regarding their efficacy.

The goal of this thesis is to add to the literature regarding liquidity risk by surveying the current approaches to its management. The aim is to distil a complete and thorough definition of market-related liquidity risk so as to clearly and objectively assess to what extent the current stock of liquidity risk measures capture the totality of liquidity risk.

Unlike previous approaches which have concentrated merely on a theoretical discussion, this thesis focuses on the measures' ability to accurately quantify the totality of liquidity risk. In this respect, the various measures and models are assessed with both a theoretical critique and empirical tests. Such tests occur within the realities of the price-formation process and are devised to determine how accurately the models hint at realized liquidity risk.

The overall objective of this dual approach is to arrive at a comprehensive representation of market-related liquidity risk and the measures used to monitor it. This is achieved in several steps.

Chapter 2 lays the bedrock of the theoretical analysis by defining the multiple aspects of liquidity risk and deriving as complete a definition as possible. Chapter 3 then showcases the importance of liquidity in market-risk and discusses how standard risk measures like Value-at-Risk (VaR) fail to account for it.

Chapter 4 uses the results of Chapter 2 and critiques the liquidity-risk approaches put forth in the literature: beginning with the standard market measures, which seem to proxy for one aspect of liquidity risk, and ending with the more integrated liquidity-risk models which try to capture the

gamut of liquidity risk. The chapter analyses each of the measures/models within the context of the definition and assigns them to groups according to organic similarities in the information they present.

Chapter 5, finally, presents the results of an empirical study into the accuracy of the risk measures and the similarities between them. Since some of the liquidity metrics are observable market variables (the so-called measures) which do not provide objective forecasts, they are assessed differently to the models, which are rigorously backtested.

The measures are assessed on the basis of their sensitivity to periods of known market-wide illiquidity with the help of long-term data. This tests whether the liquidity measures provide information on realized liquidity at all.

Next the measures are subjected to a correlation and principle component analysis (PCA) in order to assess empirically whether different measures provide different insights into liquidity risk. This determines to what extent the theoretical groupings of the measures are actually real.

Models, like the liquidity Value-at-Risk (L-VaR) models, which provide objectively verifiable forecasts are analysed more deeply. Unlike previous studies these models are implemented and carefully backtested in order to verify their accuracy. They are also subjected to a sensitivity and trend analysis. The focus is on highlighting the behaviour of the measures/models and assessing how well they perform at implementing their theoretical framework in the face of realized liquidity concerns.

The results of the analysis show that liquidity risk is composed of a variety of complex aspects, all of which contribute to the variability in the expected proceeds from liquidation. While each of the standard liquidity risk measures provide information on different aspects of liquidity risk, generally they are not exhaustive and many important aspects pertaining to liquidity risk are ignored. This may contribute to the measures' poor ability in predicting changes in liquidity.

Although the integrated L-VaR models are more comprehensive and aim at integrating all of the different aspects of liquidity risk into a single framework, they are not generally more accurate in forecasting portfolio losses than standard VaR models. Indeed in many cases some of the more sophisticated models, perform worse than models which tend to focus on only a single aspect.

The problem thus lies in capturing all of the aspects in a single measure in a



way which does not leave the model unduly dependant on any single aspect and which offers a fair reflection of a market-participant's likely loss with liquidity risks.

Although some of the less integrated models are more accurate than standard VaR techniques, the fact that they ignore certain important aspects of liquidity risk, implies that in order to properly manage liquidity risk, a market practitioner needs to make use of an array of specialised measures.

University of Cape Town

## Chapter 2

# Towards a Definition of Liquidity Risk

Risk is commonly understood as measurable uncertainty surrounding future outcomes. Unlike uncertainty alone, which is merely the inability to forecast the way events play out, risk can be modelled and quantified. Market risk measurement, in particular, is concerned with the measurement of uncertain financial market outcomes and liquidity risk with measurable uncertainty relating to liquidity-driven events.

Precisely which events characterise liquidity risk is, however, the source of much contention and only recently has more rigour been applied in defining them. This seems odd given that liquidity is widely accepted as an important determinant of market risk [39] and that general risk modelling techniques are well-established in the market.

The principle reason for the delay in applying known risk modelling techniques to liquidity risk is that there is no consensus as to what constitutes it. There has been, until recently, only a general understanding that market liquidity is related to the market micro-structure and the price formation process. Moreover although advances in market micro-structure theory have meant that the role of liquidity risk and how it arises has become better understood. There is still a lack of a sound theoretical development of liquidity-related market risk.

Most attempts at defining and modelling liquidity risk have been markedly incoherent and have centred around intuitive, some times ad hoc approaches and well-understood ideas. All of these seem increasingly inadequate given

that liquidity is actually generally well understood [39]. Indeed there is, at least, anecdotal evidence that the press, investors and policy makers can identify an illiquid asset [39]. This has, quite possibly, led to the complacent belief that liquidity risk necessitated no further research. Frustratingly there have also been few regulatory incentives to precipitate greater research.

The lack of regulatory incentives has arisen for the same reason that standard market risk models like Value at Risk (VaR) did not exist prior to the Basel Accord – lack of regulatory pressure to account for it. Goodhart [47] points out that in the 1980s there were attempts to extend Basel to account for liquidity risk but this was abandoned after a lack of consensus regarding definitions. In the face of the accord’s requirement that “banks only take reasonable steps to maintain appropriate systems for the management of prudential risk” and few general liquidity guidelines, financial institutions tended to be highly capitalized with very illiquid assets which provided greater returns.

Thus although the accord had taken great steps to prevent a systematic capitalization crises, it failed in that it misjudged the importance of a clear and final definition of liquidity risk. In effect, it can be argued that the Basel Accord’s omission may have engendered the perverse incentives and unchecked risk-taking that gave rise to the most recent economic debacle.

Notions like this have propelled greater recent interest into liquidity risk and its measurement. Unfortunately while literature regarding liquidity risk exists, such work seems broad and scattered with no comprehensive guiding framework. Such an uncoordinated body of theory constrains and fragments future research. The importance of a review of existing approaches and definitions, as attempted in this thesis, to unify and test cannot be understated.

The results of the review show that overall, the literature tends to define liquidity risk within the context of its intuitive aspects, its role in transaction costs and its general market-related characteristics like depth and breadth. All of these are inextricable from the process which drives them.

As noted by Loebnitz [65], in order to better understand these different aspects it is thus necessary to understand the price formation process or “the process by which investors’ latent demands are ultimately translated into prices and volumes”.

## 2.1 The Price Formation Process

Liquidity risk exists exactly as a consequence of market frictions and the failure of the law of one price. These frictions arise because of the characteristics of the price formation process and market architecture. Liquidity is intimately related to market structures and processes and these must be understood before the risk can be understood.

Typically markets take one of 4 forms: dealer emphasis trading mechanisms (DLR), pure electronic order book (LOB), hybrid mechanisms (HYB) and periodic call mechanisms [65].

In DLR markets, market-makers or dealers (also known as Specialists) are obliged to maintain order flow by always being prepared to bid or offer quotes on any trade quantity. They may set their quotes according to their discretion but must manage a quote for every order placed before them. Dealers trade on their own accounts and thus take ownership of any inventory they build up through trade. In this way they manage to provide immediacy even if natural traders do not exist on the other side of a trade.

LOB markets do not have traders and all orders are matched electronically according to rules of precedence [65]. Those orders which cannot be matched immediately are accumulated on a central order book which all traders can access until they are matched later in the day. The JSE, as discussed in Chapter 5 in more detail, is an LOB market with opening and closing call auctions.

Call markets consolidate orders over a period of time which are then executed at a price that maximises volume. The NYSE is such a market that starts trading with an opening call auction [65].

HYB markets use a combination of LOB and DLR mechanisms to facilitate order flow. The majority of stock markets in the world are HYB. Thus the LSE, NYSE, NASDAQ, etc. are all HYB markets [65].

All trades entered into any of these markets go through 4 distinct processes: information gathering, order routing, execution and clearing and settlement. Order routing and execution are probably the most important aspects of any trade.

Order routing is the process by which an investor places an order with his broker and then the broker, depending on the characteristics of the placement, attempts to fill it. Brokers have a choice between “best execution”,

meaning that they must find the best time and price, and filling the order from internal inventories. Many orders also have execution conditions meaning that investors can dictate provisos for their placement in the market.

Trades can be market orders, which means they will be met at the best available market price or limit orders wherein the investor specifies a price limit after which the order will not be met. Conditions include fill-or-kill or time constraints involving when an order should be met. Generally limit orders are carried on the book for a day and if not met then killed.

Irrespective of their attendant conditions, orders are always entered and then matched according to rules of precedence. Generally priority is given first to price and then to time of placement but in some exchanges larger orders which come later may have priority over the smaller and older orders [65]. This pertains to the execution process.

Beyond the above intricacies, markets also place restrictions on trades. These include trading halts, collars, margin requirements, exposure limits and minimum tick sizes. Moreover as Morris *et al* [68] note, during times of extreme turbulence, markets actually shut down. These restrictions and events render prices in-discrete as jumps only occur in multiples of the tick size, after trade halts or once some benchmark level has been regained.

Price moves in the market are thus significantly affected by the frictions which beset them. They are a consequence of subjective expectations which propagate the signed order flow that crystallizes as demand and supply. These are a function of the market architecture in which they take form. Since demand and supply are merely the combination of all the buy orders and all the sell orders at a particular point in time. They are thus inextricable from the price formation process.

Given this there seems little sense in making assumptions of lack of friction in risk modelling. One must properly account for the costs of trade which arise because of illiquidity and market friction. Ignoring them, as seen later, in the high-profile debacles of Amaranth and LTCM, can lead to grave error and underestimation of risk.

## 2.2 Intuitive definitions

The most basic attempts at defining the liquidity-related costs of trade are the popular, intuitive notions of liquidity risk. These have improperly linked

the liquidity of an asset to its ease of convertibility into cash.

Intuitive definitions seem to hint at both speed of conversion and associated costs [39]. These definitions hold cash as the benchmark liquid asset with all other assets measured against it. Many papers thus regard liquidity risk as the risk of “being unable to liquidate a position in a timely manner and at a reasonable price” [69]. The implication then is that for assets held to maturity, liquidity risk must be insignificant [16].

Under the above characterization, a perfectly liquid asset must be one which can be easily exchanged in infinite volumes and at zero cost. The notion of exchange, however, immediately conveys the idea of market-related risk and undermines the role of cash as the benchmark liquid asset, as cash bears no market-risk. Any definition of liquidity must necessarily recognise the fact that it primarily stems from market-related activities and micro-structures.

The difficulty then in using a cash-based definition of liquidity is that firstly, cash does not convey any idea of the normal costs associated with trade. In addition market-related aspects are masked and often confused with other related, but less directly significant aspects like liquidity of the economy and the firm.

Economy-wide liquidity relates to broad notions of credit extension, monetary momentum and money supply issues [42]. These are indicative of the ease with which cash is made available and are the preserve of the monetary policy authorities whose goal is to ensure monetary stability in the interests of economic growth. Firm-wide liquidity, operates at the micro-economy and relates to issues of operational stability and debt. A firm is considered liquid if it can pay its liabilities as they fall due. Firm-wide liquidity is thus associated with asset and liability matching and other internal issues.

Contrastingly market-wide or asset liquidity is externally driven by financial market events and the market-wide appetite for an asset. It is the major source of risk in the modern financial system and, as seen by recent credit-market events (and shown later in Chapter 3), crucially drives firm-liquidity and economy-wide liquidity.

Although this thesis focuses on market-related liquidity risk, as discussed in Chapter 3, all of the above facets of liquidity risk are important and related to the stability in market liquidity, particularly during periods of crises. They are, however, not central to the definition of liquidity risk.

The realization that cash has no market risk in and of itself and is merely a medium of exchange has led to more market-based definitions which focus

on liquidation value and the risks which make this lower than expected. These definitions focus on liquidity risk as “the danger a market participant faces in not being able to immediately liquidate a position at a price close to the current market price” [44]. They focus on the costs associated with this immediacy.

## 2.3 Transaction Cost-Based Liquidity Risk

The most significant aspect of market-related liquidity, are the transaction costs which are attached to trading in an environment with friction. Both Lybek *et al* [66] and Angelidis *et al* [10], and many other papers, distinguish between implicit and explicit transaction costs and cite them as the transaction cost-based aspect of liquidity risk. These are the events which characterize liquidity risk.

Explicit transaction costs relate to order processing costs, taxes like Marketable Securities Tax (MST) in South Africa and brokerage and other costs incurred to trade a security. These costs are usually associated with the communication and initiation of a trade and are largely certain prior to the trade [65]. They do not add markedly to overall liquidity risk.

Implicit transaction costs, however, do influence trade uncertainty and heighten market risk. The literature includes any events which tend to reduce the expected proceeds from liquidation under this banner. These include: the bid-ask spread, price risk, opportunity costs and price impacts [65], all of which induce uncertain effects on the value under liquidation.

### 2.3.1 The Bid-Ask Spread

The Bid-Ask spread refers to the difference between the Bid and Ask price available in the market at any given instant. The spread changes continuously throughout a trading day in so far as available bids and offers change.

The Bid price for an asset is the highest price that any buyer who is currently quoted in the market is willing to pay and the Ask or Offer price is the lowest amount that any seller is willing to commit for that asset. Generally the Bid is almost always lower than the Ask due to arbitrage whitening, thus the spread is positive.

As noted, in much of the literature the Bid-Ask Spread does constitute a transaction cost in that if an investor buys at the Ask then he may only be able to sell again at the lower Bid. If for example the Bid were \$30 and the Ask \$35, one would expect to lose \$5 or more on every such trade <sup>1</sup>. This is a paper loss and exists immediately at the time of purchase – it is not split half at the time of purchase and half at the time of re-sale.

The spread is a cost to all trades, but can be mitigated by purchasing at the ask or higher and being patient until another buyer arrives to meet that ask at a better bid. In this regard the spread rewards the patient investor who is willing to bear the risks of the market moving against them during the period between buy and sell and of their orders not being met as other orders with priority get filled first.

Generally the width of the Bid-Ask spread and the reason for its change can be attributed to divergences in Supply and Demand. Higher Bids mean relatively higher demand and a falling spread, similarly for sinking supply and falling Ask prices. The Bid-Ask spread is then intimately related to liquidity – it gives an indication, in so far as it captures deviations in demand and supply, of the expected costs if a position were liquidated. The lower the demand and the higher the spread, the greater these costs (as one would now buy at the much higher Ask and could only sell at the much lower Bid), similarly with higher supply. This point is made somewhat more clearly in the context of a Dealer market, with a fixed market-maker, than in an Order market where it is said that the “patient” investor makes the market.

In an order-driven market the spread is driven primarily by the behaviour of the market-maker or dealer, their quotes and the prices they post. Dealers are tasked with maintaining order flow in an order market and must trade continuously, sometimes from their own account. They face significant risks in that they can never be certain to whom they can sell or from whom they buy, nor at what price to trade. Despite this they must trade, as they only make profit by encouraging trade, either by way of pricing aggressively compared to other dealers or by developing relationships with their clients [65].

By trading continuously, dealers inherently face two concerns: inventory accumulation and the risk of trading against informed traders. These are the two principle reasons given by the micro-structure literature for the existence of the spread.

---

<sup>1</sup>One would pay, at best, \$35 to purchase the asset and then could only sell it immediately for \$30



The literature argues that the spread seeks to compensate the market-maker for accumulating inventory and thus bearing more market risk than other traders to facilitate trade. They must maintain some level of inventory even if the market is expected to fall. Usually, however, dealers set inventory targets and set their prices to meet these targets. Thus like any shop-keeper they raise bids to build inventory and lower asks to diminish it. Effectively then the BA spread is determined by dealers' inventory concerns and a premium for inventory-related risk [65].

Beyond this the spread may also exist to compensate dealers for losses due to informed trading and asymmetric information. If dealers feel that order flow is one-sided, then they may suspect that the trading is informed. They may then change their prices to protect themselves and moderate the flow [65].

Research into the spread initially began as a signpost into transaction costs and progressed into research into its determinants and the reason for its existence. In one of the earliest studies on the existence of the spread, Stigler [77] defines the spread as the "cost of consummating a transaction" which he hypothesises as half the spread plus any commissions. This was later extended by Demsetz [35] who defined transaction costs as the "cost of exchanging ownership titles" which was further narrowed down to include only the spread and any brokerage fees. The spread was thus thought to be the primary cost associated with liquidity.

Indeed Ting *et al* [78] argue that much of the later work on the spread was based on the notion that it is proportional to the mark-up paid by investors for immediacy. In line with this Glosten and Harris [46], decompose the spread into order-processing costs and costs due to adverse selection from informed trading. Bervas [74] points out that the spread captures order-processing costs, the volatility of accumulated order flows and the degree of informational asymmetry. It is essentially a premium paid for liquidity.

Irrespective of the reason for the spread's existence, the literature seems to agree that higher spread volatility increases the uncertainty around expected liquidation proceeds and increases liquidity risks. While the spread is certainly no longer believed to be the sole cause of liquidity risk, it is widely accepted that a higher absolute spread raises transaction costs and, by association, liquidity risk.

### 2.3.2 Price Risk

The act of trading and thus liquidation induces an additional uncertainty and possible cost in the form of price risk. It is the risk that the market will move in the opposite direction to that expected when the trade was first initiated. This risk arises solely as a consequence of the market-related moves between order processing and execution.

Price Risk is illustrated in Figure 2.1 where during the time interval  $\tau$  between order placement and final execution, the trader is vulnerable to the totality of market-movements.

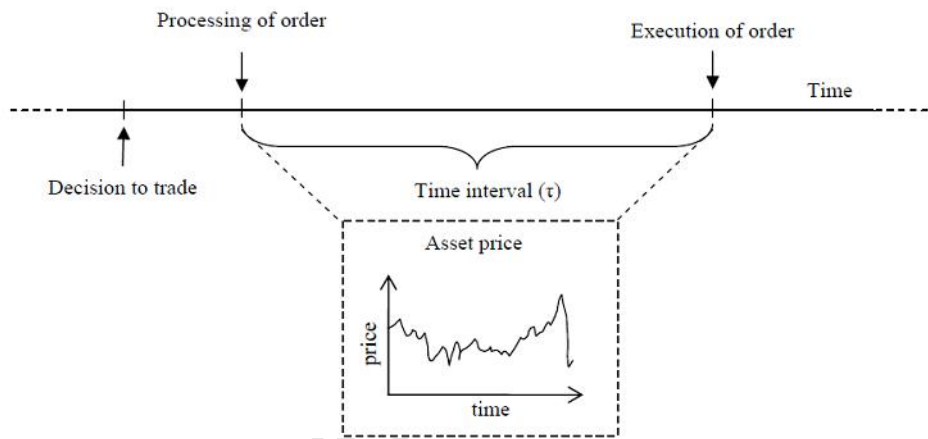


Figure 2.1: Price Risk [65].

Loebnitz [65] argues, quite reasonably, that traders expect to trade at the last transacted price or better but never worse, or else they would postpone their trading – supposing that they can defer this. Since there are penalties for expecting to buy or sell at a very different price from the last, price risk induces an uncertain future cost (or profit) in that expectations are usually not met [75]. This risk is necessarily smaller the more quickly a trade can be closed. Accordingly larger, more complicated trades which may take longer to consummate, immediately incur risks of higher liquidation costs.

Indeed all traders face a choice as to how fast their order is filled. In order to make an informed trading decision, traders must make a reasonable expectation of where the market will move prior to processing a trade. The accuracy of this estimation separates the good traders from the bad.

Upon initial inspection it may seem that any trader can reduce their price risk by trading immediately and shortening the duration of  $\tau$ . However,

this induces additional costs in the form of price impacts whereby the act of trading itself effectively moves the market away from the trader and realizes the price risk. This is especially critical for large orders which capture the bulk of the volume traded on a specific day.

Traders thus face a choice between trading immediately and minimizing price risk or trading in smaller amounts over time with lower price impact costs but higher price risk. Figure 2.2 below illustrates the increase in price risk as the time to fill an order varies. The positive correlation is striking.

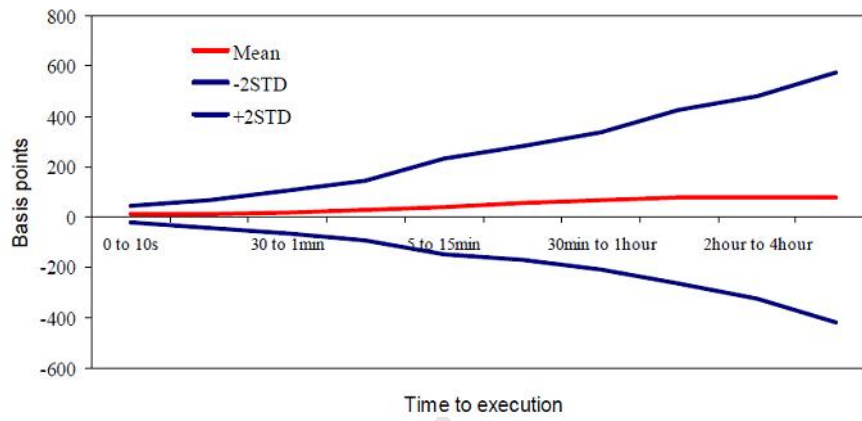


Figure 2.2: Price Risk over Time [65].

In general the decision relating to the timing and the progression of a trade is especially significant in volatile markets where price moves can be large and difficult to call. As discussed later this is a feature of markets that compounds liquidity risk during times of crises.

### 2.3.3 Opportunity Costs

The third implicit transaction-cost related to liquidity are the unverifiable but ever-present opportunity costs attached to trading. These costs are never known prior to initiating the trade [65].

Although Loebnitz [65] mentions this cost, they only go so far as to argue that it cannot be reasonably measured and that it is completely uncertain. They do not actually discuss what opportunity costs exist during liquidation but restrict themselves to a discussion of the Perold Shortfall measure. This measures the opportunity cost as the weighted amount by which some benchmark has changed during the period of trade. In this formulation price risk and price impacts are included.

$$\text{Implementation Shortfall} = \sum_{i=1}^N (n_i - m_i^e)(p_i^e - p_i^b) \quad (2.1)$$

$n_i \rightarrow$  number of shares of the  $i$ th security in the benchmark  
 $m_i^e \rightarrow$  number of shares of the  $i$ th security held at the end of the order execution  
 $p_i^e \rightarrow$  price of  $i$ th security held at the end of execution  
 $p_i^b \rightarrow$  price of  $i$ th security at beginning of execution  
 $N \rightarrow$  number of securities in the benchmark

In general opportunity costs are attached to every action. In relation to liquidity risk they arise in that every trader always faces the opportunity cost that liquidation should not be effected and that the position should be held in the market to benefit from capital gains. They also arise in relation to the speed of liquidation – slower liquidation induces greater opportunity costs from price risk [74].

The bulk of opportunity costs seem to arise from the trade-off between price risk and price impacts in liquidation. Total liquidity costs lie somewhere between the execution cost, which falls with time, and opportunity cost, which rises [69]. This is shown in Figure 2.3 with the cost lying somewhere between the “variability of execution cost and the opportunity cost of waiting to trade later” [69].

The cost remains difficult to capture and describe and is largely ignored in the literature [69].

### 2.3.4 Price-Impact Costs

A fact which a large number of market risk models ignore, but which is gaining greater prominence, is that the act of trading itself induces a liquidity cost, commonly referred to as the price impact of a trade.

In one of the first explicit models of this factor, Bangia, Diebold and Schuermann [14] argue that liquidity risk can be decomposed into an endogenous component, related directly to the size and manner of trade, and an exogenous component that is linked to market-wide conditions of liquidity. Both are significant in the context of aggregate market risk.

The exogenous component refers to transaction costs as captured by the volume, Bid-Ask spread, etc. while the endogenous component relates to trades which effectively move the spread and induce price impacts.

As noted in Bangia *et al* [14], Loebnitz [65] and Hisata *et al* [49], any trader who enters the market with a position that is larger than the quoted size available, can only fill the order by moving down the order book and accepting poorer prices or trading outside the quoted spread. The alternative is to partially fill the order at prices within the spread and fill the rest outside the spread or to kill the order and wait for larger quote sizes. Bangia *et al* [14] clearly represent this scenario in Figure 2.4.

As shown once the trade size is outside the quoted depth or available size, prices become increasingly prohibitive. Indeed the price at which to liquidate decreases at an increasing rate.

Loebnitz [65] argue that the size of the price impact depends on the benchmark from which one computes it. They point out that the literature distinguishes between pre and post-trade price impacts.

Pre-trade price impacts refer to the “adverse deviation of the actual transaction price from a benchmark price”. In this regard the benchmark price could be what the paper refers to as the “base price”, that is the price of the asset had there been no price impact [65].

They illustrate the pre-trade impact rather clearly with an example of a

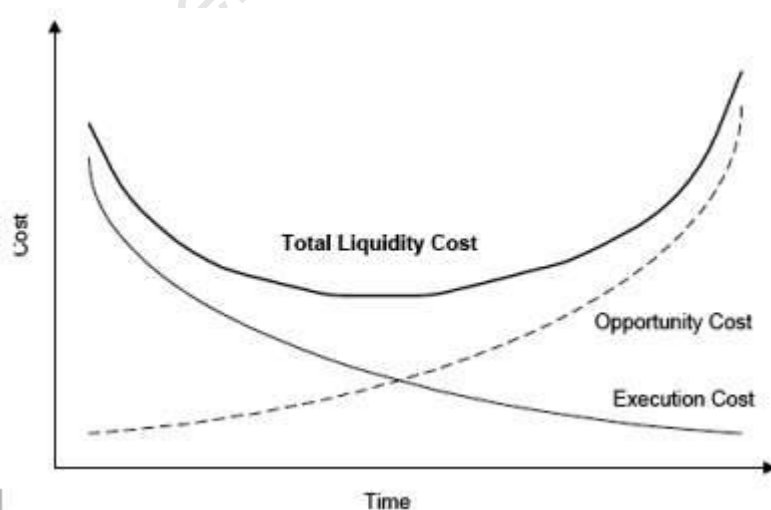


Figure 2.3: Opportunity Cost Trade-Off [69].

trader who wants to trade an order of 600 000 units in 3 equal splits over a period of a few hours. The trading is indicated below in Figure 2.5.

In the example they give, a trader decides to buy at 09:45 when the Ask is \$25.4. As the order takes time to fill and as the market learns of its size some front-running occurs and Asks rise in anticipation of a desperate

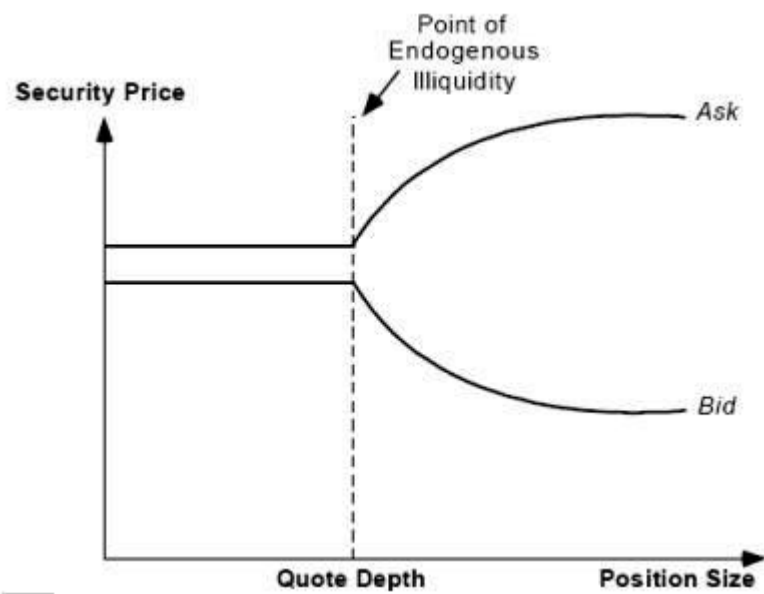


Figure 2.4: Price Impacts [69].

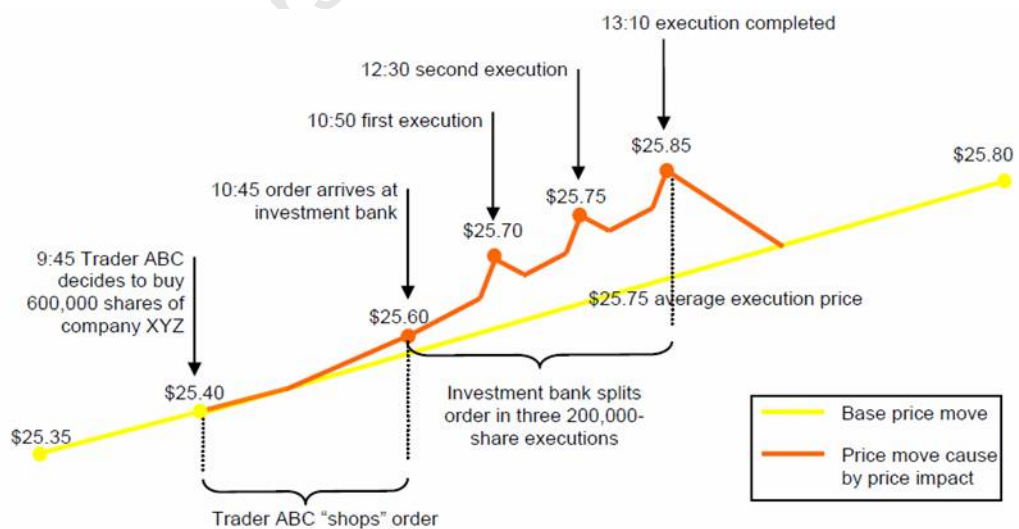


Figure 2.5: Pre-Trade Price Impacts [65].

buyer. The Ask continually rises so that by the time the order is closed, the average cost of the transaction is \$25.725, which is a 1.27% premium to the base price of \$25.40. In this case, relative to the base price there is a price impact.

As noted in Loebnitz [65], if there were no liquidity costs, all transaction costs would occur at the base price and there would be no price impacts. In this case, assuming that the base price rises over time and does not fall – it could just as easily have fallen over a day – then it would be most optimal to institute a block trade at \$25.40. Interestingly, using arbitrage arguments, Çetin, Jarrow *et al* [23] rigorously prove this by assuming a stochastic supply curve that is increasing in size. Block trades are most optimal if there are no liquidity costs.

Loebnitz [65] define the post-trade price impact as the price impact which occurs when the actual transaction price occurs within the spread or even at the Bid or Ask. When this occurs and there is a price impact after the trade then there is a post-trade impact. Generally the post-trade impact is less practically important to traders than the pre-trade impact as the latter describes expectations and real costs. The post-trade impact is only marginally interesting when transaction data is unavailable [65]. The argument below distinguishes between the two price impacts.

Let  $t_0$  be now and  $t_1$  be an instant later when the bid/ask price has changed. Then the post and pre-trade price impacts are given by:

$$\text{Post-Trade impact} = S_1/S_0 - 1 \quad (2.2)$$

$$\text{Pre-Trade impact} = \tilde{S} - S_0 \quad (2.3)$$

$S_0 \rightarrow$  mid-price at time  $t_0$  just prior to the trade  
 $S_1 \rightarrow$  earliest mid-price after the trade at  $t_1$   
 $\tilde{S} \rightarrow$  the actual transaction price

In the definition of the pre-trade price impact the mid-price is used as the reference price. The mid-price is the most common benchmark price, but alternative benchmark prices,  $V$ , can be used. These would result in very different measured price impacts. In general to accurately measure the price

impact one would need to determine the base price. This is the price which would have been quoted had the trade for which we are calculating the price impact not been transacted.

Identifying the base price away from the transaction price in this way, allows one to separate out “normal” market-price movements – that is price risk – away from liquidity-induced movements and allows for more accurate estimation of the liquidity transaction costs. Attempts have been made in the literature to do just this: Chan and Lakonishok [25] and Patel [71] with the use of a correction term. These attempts usually take the form of a regression model (like that in Equation 2.4) which proxies the base price movement by the movement in some aggregate market index.

$$\text{Pre-Trade impact} = (\tilde{S} - V) - \beta(\tilde{M} - M_0) \quad (2.4)$$

$\tilde{M} \rightarrow$  index price at time of execution  
 $M_0 \rightarrow$  index price just prior to execution  
 $\beta \rightarrow$  the slope of the regression line

Naturally such methods are prone to much estimation error and are necessarily difficult. The question is whether such a separation is possible or even desirable. Loebnitz [65] argues that such a separation necessarily requires an understanding of what drives the base price and how it changes. This would help apportion price movements to price risk and liquidity risk respectively.

As discussed in Section 2.1, the manner in which prices form implies that price movements are determined primarily by aggregate order flow. Aggregate order flow drives the base price movement and the single order size relative to this aggregate flow, determines the trade price-impact. Separating out price risk from the price impact then amounts to separating out the effect of a single order on prices from that of the host of orders going through the market. The sheer scope of such a filtration makes the task almost impossible.

Moreover from a risk perspective, while it may be useful to isolate the price impact risk induced, which the trader can ostensibly control, from that of price risk, such a separation is unnecessary. In general traders are interested in aggregate risk not the decomposition thereof. This is especially true as it is unlikely that any trader can completely control price impact risk by trading more efficiently.



Given the difficulty of a complete separation, the majority of liquidity risk models, building on the work of Bangia *et al* [14], subsume the base price movement into the price impact by decomposing the price impact into an endogenous component and an exogenous component which includes price risk.

The exogenous component reflects variables that change the base price: aggregate order flow, dealers' behaviour, information flows, dealers' inventories, etc. It is common to the market and is usually attributable to shocks that impair expectation formation [60] and spread behaviour [44]. The endogenous component, however, reflects trade-unique characteristics like the traders' identity (directors' dealings carry more information in the market than an unknown trader for example) and the order size [65]. The size of the order relative to the norm of the market is especially important [66].

Generally the size of the price impact would depend on whether it is dominated by endogenous or exogenous factors. Indeed much of the literature argues that price impacts have a temporary aspect which is attached to the endogenous component and a permanent one that is driven by exogenous, market-wide factors which are sensitive to the volume traded. Holthausen, Leftwich, and Mayers [50], Hisata and Yamai [49], Almgren and Chriss [7] and many others argue in favour of such an incidence and a wide array of papers have documented this effect in equity, forex, bond and futures markets [49].

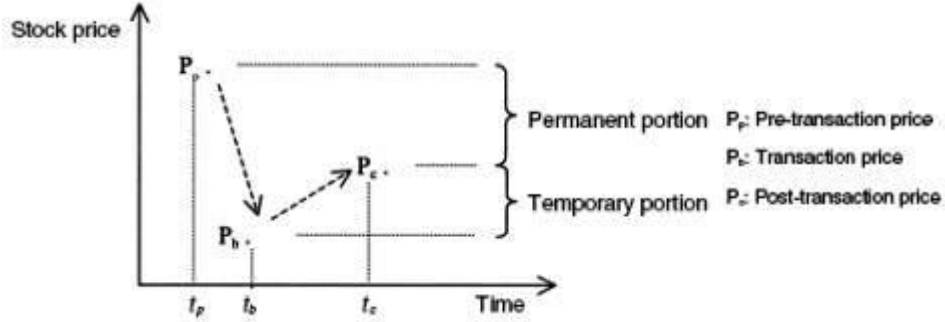


Figure 2.6: Temporary vs Permanent Price Impacts [49].

The literature attributes the temporary impact to inventory control effects, price discreteness (prices are not infinitely divisible but have minimum tick sizes), price-order pressure, order fragmentation and price smoothing. The permanent impact is attached to asymmetric information and informed trading. Although both of these can be attributed to endogenous and exogenous components, unlike price risk which is somewhat independent of the trade

<b>Market</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
Bid Price	Thin & Shallow	Thin & Deep	Broad & Shallow	Broad & Deep
\$50	100	100	500	500
\$49	200	200	500	500
\$48	0	300	0	700
\$47	0	300	0	900
\$46	0	300	0	1500

Table 2.1: Depth vs Breadth

size and more indicative of market risk, they are dependant on the volume traded relative to what the market is trading at the time of trade. Even though the two notions are not the same, the literature merely models the price risk together with the permanent price impact.

Modelling price-impacts is thus not a trivial exercise and the literature has found a number of characteristics which must be accounted for in their modelling.

Firstly, buyer-initiated price impacts tend to be larger than seller-initiated ones [50]. Moreover, for many markets there exist serial return dependencies as the permanent, information-based impact tends to be lagged and distributed over time [65]. Price impacts also seem to become more severe, the larger the trade size is in excess of some threshold. This is possibly because larger orders are suspected of carrying more private information [65]. This becomes particularly evident in times of crises when price impacts tend to spike and makes accounting for liquidity risk fairly complex.

## 2.4 Other Aspects of Liquidity Risk

Liquidity is further complicated by the fact that it is not completely defined by transaction costs alone. While the above transaction costs capture the bulk of liquidity risk, liquidity is characterised by a few other factors which impact on the market as a whole. Although their effects, in the end, become evident as liquidity-induced transaction costs, they are tied more generally to economy-wide and market-wide effects.

Beginning with Kyle in 1985 [59] (and even earlier with Black in 1971 [19]) liquidity has been associated with notions that impact on the elasticity of demand and supply. These characteristics like depth, tightness, resilience, breadth and immediacy all feature prominently in market practitioners' descriptions of liquidity [65]. They are illustrated in Figure 2.7 and Table 2.1.

The number of shares available for trade at a particular price defines depth [75]. In Table 2.1 the values  $OA'$  and  $OA$  define depth [74]. The more Offers there are at a particular price the greater the depth. This is negatively impacted on by higher transaction costs [66].

Breadth refers to the number of orders available and their size. A market like Markets 3 and 4 in Table 2.1 are broad as they will induce smaller price impacts for larger orders [66]. The distinction between breadth and depth is more easily understood from Table 2.1, where it is easily seen that Market 4 is far more liquid than any of the others.

Market 4, may have breadth and depth and yet lack Resiliency, however. Resiliency loosely refers to how easily a market can absorb shocks and rapidly revert to “efficient pricing” [74]. In the face of a shock where prices change rapidly, a less resilient market will stay illiquid for longer. Lybek *et al* [66] refer to this as the tendency of markets in which “new orders flow quickly to correct order imbalances which tend to move prices away from what is warranted by fundamentals”.

Much of the literature also makes mention of Immediacy and Tightness, intuitively arguing that a liquid market should have low costs to turn a

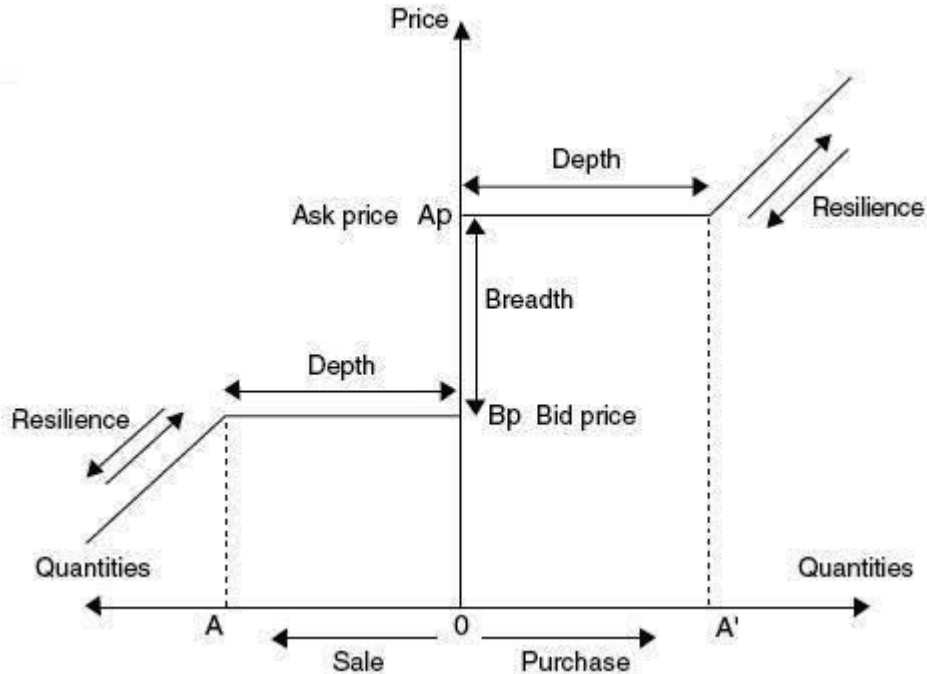


Figure 2.7: Aspects of Market Liquidity [74].

trade around (Tightness) and should promote quick entry and exit of trade (Immediacy) [74]. These characteristics vary directly with the number of willing buyers and sellers in the market at any time [66].

All of the above aspects of liquidity seem to allude to ideas of elasticity of demand and supply. Indeed it seems natural to consider market liquidity from the perspective of elasticity, as the more elastic a market's demand or supply, the lower the price impacts of trade. Indeed many of the earlier papers: Grossman and Miller [48] and Kamara [57] argue that liquidity risk only arises as a consequence of downward-sloping demand curves which is guaranteed so long as there is asymmetric information between buyers and sellers (for example in the classic lemon's problem where the more an asset is sold, the more suspicious the market becomes of it and the lower the bids) [16]. Thus the idea of elasticity is fairly solidly entrenched and has propagated the later ideas around depth, breadth, etc.

Elasticity of demand and supply captures the common notion of liquidity: a market is liquid if it has a wide diversity of market participants with heterogeneous expectations and trading behaviours [65]. This accounts for its wide appeal and common use. It is merely one aspect of the many that characterises market liquidity.

Many may argue, however, that a discussion of these additional aspects of liquidity seems redundant given that in the end all of their effects are captured by transaction costs, whether implicit or explicit. The precise and clear exposition allowed by the transaction cost model seems to capture all that is important.

The principle objections given to the use of concepts like breadth, depth, etc. is that they overlap and that the data used to measure them do not correspond neatly to one particular dimension (for instance volume traded which commonly measures breadth is also indicative of market depth) [66]. This complicates accurate definition and measurement. Loebnitz [65] for example points out that by "defining [liquidity] as the totality of the above, one cannot separate out one aspect from others". However they also concede that there seems to be no escaping the multi-dimensionality of liquidity.

Overall the other aspects convey important descriptive information. Not only are they important from a historical perspective, they act as descriptive tools for the liquidity of a market in aggregate. They help shift the focus of liquidity risk from the effect of a specific trade in a market to its general features.

The broader aspects thus allow policy-makers, would-be investors and other

parties to assess broad notions of liquidity and modify their decisions based on them. For example knowing that a market lacks breadth, can prepare traders for the higher price impact costs (relative to other markets) which are likely in that market without forcing them to first calculate the price impacts attached to a particular trade.

In short the additional notions, although they can be subsumed into the transaction cost basis, add richness to the inherent multi-dimensionality of liquidity. Measures which proxy for these aspects should thus not necessarily be abandoned in the face of integrated models. They provide useful general information.

## 2.5 Liquidity under Stress

The many-faceted nature of liquidity is compounded by the fact that liquidity tends to behave differently during market crashes and periods of stress. If a perfectly liquid market is one in which an asset can be bought and sold immediately at a price close to the last traded price [75] then liquidity during a crash is the antithesis of this.

Market liquidity “depends solely on the number of participants and their willingness to trade ... [which] in turn depends on investors’ expectations of price developments, the information available and their risk aversion” [74]. The fact that this is cumulative in nature means that the size of the investor base – the market breadth and depth – greatly influences it [74]. Liquidity is extremely fragile.

Indeed even with a broad investor base, the collective market psyche and heterogeneous valuations and expectations that drive market equilibria can be subject to severe dislocations which rapidly impair expectation formation and drives liquidity away. Liquidity is thus “paradoxical in nature” [74] as an asset or market is never liquid in aggregate but only for the individual.

Keynes noted that “...the fact that each investor flatters himself that his commitment is liquid (though this cannot be true for all investors collectively) calms his nerves and makes him much more willing to run a risk” [66]. In this regard liquidity is extremely susceptible to the “irrational ghosts and spirits” which Keynes held to drive markets during crises.

In fact there seems to exist a self-perpetuating cycle between market crises and illiquidity. Similar to the manner in which even the threat of default

can precipitate a bank run, the whisper of illiquidity can force participants to exit en masse and precipitate a market collapse [74].

Fernandez [42] points out that rising substitutability and concentration of financial instruments has led to an increased concentration of liquidity in a few heavily-traded instruments and greater risks of liquidity crises. This threat is compounded by the rising similarity in risk trading systems, investment horizons, and risk preferences which has led to greater integration and increased correlations between asset classes and geographical regions.

Increases in technology and the resultant greater homogeneity in expectations due to greater access to similar information have also played a pivotal role. In some ways then reduced transparency can actually improve liquidity [42].

Persaud [65] largely agrees that greater consolidation is contributing greatly to increased liquidity risk. He argues that a collapse in information costs and rising risk model similarity, driven by uniform regulatory requirements like Basel II (the growing focus on VaR is a good example of this uniformity of mind) is driving this by reducing the diversity in markets. Reduced diversity in turn fosters greater positive-feedback trading and irrational price declines.

Unlike standard, gradual periods of price declines, linked to economic slow downs and the incorporation of new information, market crises which are linked to liquidity risk show more rapid declines, greater one-sided order flow and greater volatility. They seem largely irrational as prices decline uniformly across asset classes and order flow is initiated largely by sellers. Indeed the price decline is rapid principally because of the lack of counter parties. Liquidity-related crises do not have people with contrasting views stepping in at the other side of a trade to alleviate the price pressure. This is the reason for the more co-ordinated decline.

As a result during a market crises price impacts tend to become unusually large and the cost of trading large positions in a short period spikes [79]. Moreover heightened market volatility means that traders cannot readily exit their positions at reasonable prices. This results in panic selling where positions are exited at any price merely to cut losses. The fact that a single large player's endogenous risk can impact on all other players [60] leads to a convergence of falling asset prices and falling liquidity that effectively freezes markets and drives prices far from fundamental valuations.

Severe illiquidity is at base level the result of a co-ordinated decline in confidence and risk appetite driven by rising similarity in expectations. In times of market crises, notions of liquidity shift from that of a single position to

that of the market as a whole. This is where the concepts of economy-wide and firm liquidity start interacting with market liquidity.

In times of crises, exogenous risk compromises a more significant component of liquidity risk and spread volatility, etc. becomes more crucial. This is when broad, descriptive measures like breadth, etc. become particularly useful as models which function well during normal periods fail [79].

A strong case thus exists for notions and models of liquidity risk that change during times of stress – something which almost none of the models presented in the literature address. This is problematic as model failure is endemic during market crises because standard parameter estimation mostly ignores rare events. It is thus even more crucial to be aware of liquidity risk during such periods.

## 2.6 Conclusive Definition

Given the understanding that liquidity is a multi-dimensional characteristic of markets the task remains to distil a tractable and complete definition of liquidity risk.

In arriving at such a definition one must, as stated, bear in mind that liquidity has many different, inter-related aspects: firm-wide, economy-wide, market-wide and asset liquidity. Choosing at which level to focus attention and (as we have discussed with market crises) when this attention must shift from asset to encompass the market and even the economy as a whole is a model-based consideration. Deciding whether to use a pure transaction-cost based definition versus more general aspects like elasticity, breadth and depth is also an aspect of model design.

All models must, however, teeter precariously between absolute realism and completeness, simplicity and tractability. This is the inherent tension that besets modelling and is probably the reason that most liquidity-risk models tend to focus only on the transaction-cost aspect.

As shown in the model discussion and overview, focusing solely on liquidity-induced transaction costs provides a clean and clear framework from which to tackle liquidity-related problems. While, at times, difficult to implement, transaction-cost based models are intuitive and far less onerous than other models.

Elasticity definitions of liquidity risk for instance suffer from an inherent problem in that they necessarily exclude many liquidity-induced transaction costs like brokerage and quantifiable market impacts. Defining liquidity risk as the risk that market elasticity will move in a prejudicial manner seems largely nebulous. Elasticity is descriptive and does not capture the totality of liquidity. The same holds true for all the other aspects like depth, resiliency, etc.

Descriptive definitions do not capture the necessary “bottom-line” information of what amount is actually at risk of being lost (for instance like VaR) nor are they particularly tractable to mathematical modelling. Aspects like depth, etc. overlap and are thus difficult to isolate mathematically in a model.

Transaction-costs, however, have been shown to be much more amenable to modelling. As a consequence most liquidity-risk models, either implicitly or explicitly define liquidity risk as: the expected price concession required to convert an asset to cash immediately [65]. Indeed Loebnitz [65] prefers a more exact definition: “Market liquidity is the discounted expected price concession required for an immediate transformation of an asset into cash ... under a specific trading strategy”. This expressed in Formula 2.5.

For a given trading strategy  $k$  and total order size  $Q$ , the expected price concession  $C^k(Q)$  is given by:

$$C^k(Q) = \sum_{i=1}^N q_i TP_i(q_i) e^{-r(t_i-t_0)} - QV_0 \quad (2.5)$$

$q_i \rightarrow$  order size traded at time  $t_i$   
 $TP_i(q_i) \rightarrow$  transaction price of trade  $i$  at time  $t_i$   
 $V_0 \rightarrow$  benchmark price at time 0  
 $Q \rightarrow$  total position size  $Q = \sum_i q_i$   
 $N \rightarrow$  time horizon for the order execution  
 $r \rightarrow$  the relevant discount rate

Certainly while laudably precise, the definition also has drawbacks that stem from its narrowness. Although Loebnitz [65] claim that it is objective, the dependence of the definition on a specific trading strategy, means that it is really subjective. Any measure of liquidity based solely on this definition would not be particularly helpful in decision-making as it would necessitate that the expectation be computed across all possible trading strategies prior



to any single strategy being adopted. The definition does not help one identify such an optimal trading strategy.

Related to this, the definition does not provide general market information. It is unclear ex ante how a market would be considered to be more liquid relative to another. It seems again one would need to either consider all possible trading strategies or the optimal one, which is not always realizable in practise. The definition then does not lend itself easily to the different notions of liquidity: economy, firm and single position.

The Loebnitz [65] definition while precise and ideal for a single position, unduly constrains the definition of liquidity risk to unexpected transaction costs relative to some undefined benchmark. It ignores the changes brought about by a change of focus to the broader market and, significantly, does not explicitly account for market crises. This neglect is ironic given the fact that the paper concedes that the multi-dimensionality of liquidity cannot be escaped. They even agree that the ideal definition would capture all aspects of liquidity.

While it seems true that these different aspects tend to confuse ideas and hinder mathematical modelling in the area, a definition should not be so narrow so as to miss out on broad descriptive information which would be especially important in times of stress. The “other aspects” of liquidity risk provide crucial information relating to the size and scope of liquidity-induced transaction costs, as all of these costs are influenced by them. Any model must incorporate them in order to balance completeness with tractability.

In the interests of a complete and mathematically amenable model of liquidity, it seems most reasonable to define liquidity risk as: The risk of future loss in value faced by a market participant due to unforeseen changes in the expected liquidation value of an asset.

In this way the focus is changed from the size of expected price concessions to the variability thereof. Moreover whether the liquidation value is measured in cash based on a specific trading strategy or on any other instrument proceeding from any strategy, the risk of loss in “end value”, once all trade has ended, is captured.

The definition includes all associated transaction costs like price impacts, the bid-ask spread, etc. and can be extended to characterise market-wide liquidity events and market crises in so far as these impact on the unforeseen changes in liquidation value. Unlike the previous definition an asset held to maturity also has liquidity risk – such an asset will always have a liquidation

value and whether this is realized or not via a deferred or immediate trading strategy is immaterial.

Any confusion relating to which events characterise liquidity risk is resolved by the above definition as all events, like transaction costs, etc. which add to the variability in an asset's liquidation value necessarily add to that asset's liquidity risk. Assessing liquidity risk in this context merely amounts to applying standard risk modelling techniques to account for the variability in an asset's liquidation value.

University of Cape Town

## Chapter 3

# The Importance of Liquidity-Risk Management

The above exposition of liquidity risk has shown that the common, intuitive definitions which were previously used to characterise it were inadequate. Liquidity is complex and requires careful analysis of its separate components before arriving at a holistic and practicable definition. There is as much difficulty involved in arriving at a tractable definition of liquidity risk as there is in modelling it accurately.

It is this inherent difficulty that requires some motivation of why liquidity risk should even be considered for modelling. There is little sense in modelling merely for the sake of it. As shall be discussed there are many, pressing reasons for accurate and workable liquidity risk models and measures. Not only are these models needed to capture liquidity-induced transaction costs, which can be significant, but liquidity must be accounted for to maintain model accuracy in the face of liquidity-induced systemic crises; the extent of which is usually underestimated by current risk systems.

The importance of liquidity is addressed in this Chapter from 3 angles: the size and significance of liquidity transaction costs, the effects of a systemic liquidity-induced crises and the need for more accurate models to account for such events. Two case studies: LTCM and Amaranth, which were both caused by risk system failure to account for liquidity, are also discussed.

### 3.1 The Magnitude of Transaction Costs

Liquidity-induced transaction costs can be large and costly to ignore within a decision-making risk context. A number of empirical researchers have examined the size and significance of these costs and nearly all agree that the costs are large.

Domowitz *et al* [37], for example, examined the costs of institutional equity trades at a country level. By separating costs into implicit and explicit components they find that total costs account on average for 71 basis points across 42 countries. Roughly the explicit costs accounted for 2/3rd of this. Implicit costs were, however, measured relative to a benchmark and are thus only an estimate. Despite this the results are compelling: out of every 100c traded, on average 0.71c is lost to liquidation costs. On a R1m trade then, one can expect to lose R7100, during a calm market interval. When markets are volatile this could triple, moreover, as shown in Figure 3.1, this cost depends crucially on the order size relative to the volume traded on that day. During a particularly illiquid day a small order could easily capture more than 25% of the market volume and the liquidity costs could be as much as 1.68% of the value traded.

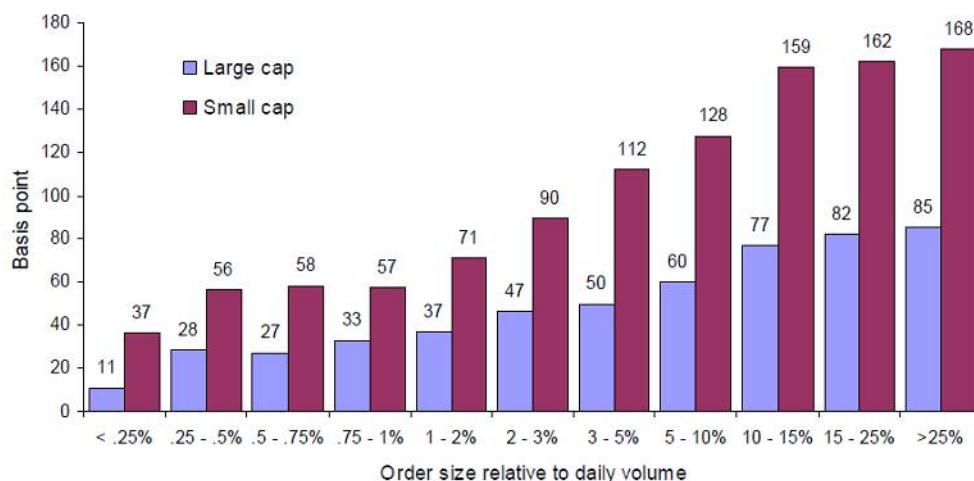


Figure 3.1: Price Impact Costs [65].

Liquidity costs also impact on other risk issues and induce knock-on costs. In the case of dynamic hedging for example, it is well known that as a result of market frictions not all assets can be perfectly hedged. Thus many assets are only approximately hedged and open to price exposure. François-Heude *et al* [44] point out that ignoring liquidity costs when hedging can

underestimate market risk by as much as 25-30%. They note that liquidity-related costs can be as high as the costs involved in adverse market price movements. This is particularly true of volatile markets in times of crises.

### 3.2 Liquidity, Solvency and Systematic Crises

During times of crises, liquidity price impacts rise dramatically and the inter-relationship between price risk and liquidity risk changes. This has been indicative of the major market crises of the 21st Century where illiquidity risks in a few large players has brought them close to insolvency and then spread out across the markets.

The perplexing link between a firm's solvency and the market liquidity of its assets, coupled with systemic crises across asset classes has prompted many researchers to take liquidity risk far more seriously. The IMF has argued that market illiquidity is the single most important factor that has triggered the preponderance of market crashes across many markets [60]. Liquidity is thus crucial for the prevention of financial meltdown.

Among the papers which have struggled with clarifying the link between solvency, economic crises and market liquidity, Acharya *et al* [1] provides the most extensive and understandable analysis. They examine the dual role of the modern financial intermediary as a provider of capital and as an investor of capital and the inherent conflict in times of crises that this engenders. This is particularly true of firms whose primary source of profit is directly market-related: brokerage firms, specialists, trading desks at large banks and hedge funds.

Like all other firms, such entities face funding liquidity risk or firm liquidity risk, being the risk that internal firm cash shortfalls cannot be funded timeously. Unlike other firms, however, the risk of such cash shortfalls in these institutions is also market-related and significantly driven by market prices. Mark-to-market cash needs, collateral obligations, hair cuts, etc. are all subject to market risk.

All trade requires capital and in institutions which trade as their main activity of business; this capital is either borrowed from cash flush entities like banks in the form of leverage or from privately-raised equity capital. Trading firms take this capital and invest it in a range of opportunities which they hope will earn them profits high enough to cover the cost of borrowing. In this way they earn revenue for their investors.

Generally the borrowed capital will be in the form of direct borrowing, which lies on the entity's balance sheet, but as has become more common, can be in the form of leverage trades, collateralized debt obligations (CDOs), bonds, haircuts or underwriting, all of which incur a cost. Mostly this cost, as in the case of futures, some derivatives or other assets which are marked-to-market, require cash flows which vary directly with the value of the investment made. In these trades cash flow needs are market-related.

Given the preponderance of these type of marked-to-market assets many financial institutions are increasingly finding that not only is their revenue market-related but so are their cash flow needs and expenses. This becomes particularly problematic in times of stress and provides the crucial link between market-wide illiquidity, firm solvency and systematic economic crises.

During times of severe price pressure and market dislocation, irrespective of the cause, the majority of asset prices fall as the market reacts to news and risk aversion rises. Falling asset prices reduce firm profits and asset values and bring a firm closer to its collateral and capital requirements. This is especially true for banks and leverage institutions like hedge funds. Banks must keep a fixed portion of their assets in cash, thus falling asset values automatically require them to reduce their asset exposure and shore up cash to meet regulatory requirements. Hedge funds face a rising cost of capital and mark-to-market calls (especially if they are on the wrong-side of a trade and the market is falling).

Faced with profit pressure and pressing cash flow needs, these institutions are forced to cut their positions and engage in fire sales. They effectively sell into a downward market to meet their individual cash needs. However as order flow becomes more one-sided [68] during crises, these institutions need to make larger price concessions in order to trade. The illiquidity of the market, propelled by a temporary shift in market-wide expectations, then propagates a market fall. In this way a feedback loop is created between an individual firm's cash flow needs, market prices and market illiquidity.

Although institutions can raise cash from other sources, the rising risk aversion associated with a market crash means that different cash avenues close. During normal periods an institution can use subordinated debt, undrawn credit lines, or, if lucky, retained earnings to fund their cash needs. However, during large external shocks, debt becomes more expensive and collateral requirements more stringent. Institutions are thus forced to sell their assets at deep discounts [1].

Acharya *et al* [1] thus conclude that large downward shocks in asset prices will always be preceded by large liquidity shocks as funding needs become

more pressing. Moreover they argue that if a single firm is large enough to have exposure to many and varied trades and is then quickly and unexpectedly faced with insolvency due to a market decline, the consequent knock-on counter-party effect as a consequence of bad debts to other firms can precipitate a frighteningly large economic crises. Counter-party risk spreads the insolvency around. In the words of Buffet “It’s not just whom you sleep with, but also whom they are sleeping with”.

In line with this Shah *et al* [74] argues that the link between market liquidity and firm solvency has been propagated, especially in the most recent crises, by rising transferability of risks between counter-parties and concentration in trading. Economies have increasingly relied on the market for allocating risks through the economy. The marketability of structured products like securitized units, etc. for example relies on the assumption that the market will find a suitable counter-party for their risks. The sudden disappearance of a counter-party has meant that the risks attached to an asset are carried over many times throughout the economy.

Liquidity is thus far more intimately related to systematic economic crises than previously believed. This has been borne out by almost every major market crises of the past century: May 1970 - Penn Central Commercial Paper crises, Nov 1973 - Middle-Eastern Oil crises, Oct 1987 – Stock Market crash, August 1990 - Iraqi Invasion of Kuwait, April and Dec 1997 - Asian Crises and June - Oct 1998 - the Russian default and collapse of LTCM. The most recent events of Aug 2008, culminating in the collapse of Bear Sterns, is the paragon of these examples.

The correlation between negative asset returns and illiquidity is unmistakable. Indeed Acharya *et al* [1] show that the innovations in liquidity (being the error term in an auto-regression of some liquidity proxy) are highly episodic, generally small but prone to sharp, dramatic upward spikes which are closely associated with systematic crises.

Evidence of this dramatic knock-on effect has prompted Acharya to divide markets into 2 regimes: a normal regime and an illiquidity regime. The normal regime is characterized by relative stability and correlations that reflect pricing fundamentals. The illiquidity regime is, however, characterised by rising asset and cross-market correlations caused by homogeneity in expectations and fear-based trading. In an illiquidity regime, returns reflect the rising cost of capital and not expected future profits [1].

Understanding how liquidity changes over time and in response to certain events can thus help market players avoid taking on silent risks which may bring them closer to insolvency. Given the link between market liquidity and

price stability, it is crucial that liquidity be accurately monitored. Indeed the IMF states that markets need to be liquid to improve their efficiency, maintain macro-economic stability and allocation of resources. Markets that are illiquid do not transfer risks efficiently and loose transparency [66]. Liquid markets are also amenable to policy intervention and control as they respond more quickly to regulatory changes.

Although by now the case for accurate accounting of liquidity in all risk measures should be quite comprehensive, the ideas relating to the relationship between funding liquidity and economic crises are best illustrated with a case study.

### 3.3 Liquidity Crises Case Studies LTCM and Amaranth Advisors

*“With market and credit risk, you could lose a fortune. With [funding] liquidity risk, you could lose the bank!” [1]*

#### 3.3.1 Long-Term Capital Management

Many individuals were surprised by the collapse of LTCM and the mounting evidence that severe market illiquidity seemed to be the cause of the firm’s demise.

Long-Term Capital Management was founded by John Meriwether, a specialist trader who left his previous employment at Salomon Brothers to start his own hedge fund. The fund specialised in high-yield and relative value convergence trades, taking advantage of the mis-pricing between assets and using leverage to gear up small profits.

The firm used a great deal of leverage to place effectively the same mispricing trade across assets, countries and markets. All of the trades were large bets on yield convergence and engendered spectacular success.

By December 1997 the fund had over \$5bn in equity and employed a host of renowned academics, some of whom were Nobel laureates, to find mispricing with carefully primed models. Such was the firm’s reputation that its balance sheet stood at \$125bn of assets, despite a gearing ratio in excess of 25:1. Off balance-sheet financing nearly tripled this to over \$1.25trn.



Much of the financing consisted primarily of repos and bank debt and many of its lenders, having faith in the firm's reputation, provided funds at close to zero haircuts and with no preliminary risk assessments. The firm, itself, was surprisingly solvent with secured debt of \$900m and 3 year lock ups on investor equity.

Despite this its positions were huge. LTCM had exposure to nearly 2.4% of the global swap market of \$29trn and even though many of these trades were offsetting making net exposure low, gross exposure remained staggeringly high. The firm failed to realise that during times of crises when markets become very volatile it is gross exposure that matters. During times of distress markets generally break historical offsetting correlations and each leg of a net trade effectively becomes directional.

Given this, in June 1998 a downturn in mortgage-backed securities led to a loss of 16% of capital. Later the Russian default and run on the Ruble, caused the firm to lose \$550m on 21 August 1998. By the end of that month, failing to cut back on its exposure, the firm had lost 52% of capital and its gearing stood at 55:1.

Although the firm felt that it could ride out these losses and was securing more capital, by September 1998 it faced a margin call on a Treasury-bond future that ironically led Bear Sterns, its prime broker at the time, to call for increased collateral. The firm thus faced a squeeze in funding risk and liquidity risk.

As markets fell, firm reserves fell. In the face of mounting margin calls the firm had to sell its huge exposures into volatile markets. The price impacts of these trades led to a greater price spiral and the firm lost more.

Since the counter-parties to LTCM's trades had required so little collateral, default would cripple many large organizations. The lack of capitalization in the fund meant that a firm liquidation would require the unwinding of billions of dollars of trades and render many large institutions vulnerable. The potential aggregate loss was so large that on 23 September, the US Federal Reserve stepped in and organized a bail out.

According to Jorion [56] LTCM failed primarily because it failed to manage its funding risk in the face of rising market illiquidity. The firm misunderstood liquidity risk and believed that its large exposures were easy to cut, when in reality price impacts were large enough to induce insolvency.

### 3.3.2 Amaranth Advisors LLC

In September 2006, Amaranth Advisors, a Greenwich-based \$9bn hedge fund went bankrupt in 3 days, losing close to half its capital on natural gas futures [27].

Even though the firm produced quite advanced VaR reports with sensitivity and concentration numbers which were monitored daily, none of these were effectively enforced. The fund had no concentration and no stop limits. As a consequence the fund became dominated by one trade: short summer and long winter gas contracts.

Although the trade was highly successful and was implemented by highly successful traders, it was generally not realized that trade exposures were simply too high. Indeed in response to a loss of \$696m on 7 September the firm decided to double up.

Trading sentiment rather than detailed rationale seemed to drive decisions at the firm. The managers attributed the fund's failure to a series of unpredictable events [27]. However, NYMEX officials repeatedly asked the firm to curtail its exposure. The firm responded to these threats by shorting additional contracts to reduce their net exposure. Once again they failed to realise that net exposure is not the driver of risk.

Chincarini [27] re-calculated the Value at Risk of the firm based on their actual positions and find that for an unleveraged position of \$10.22bn, the predicted VaR loss was \$391bn. However had these positions been held to end September the actual realized loss would have been \$629.97bn. The paper also finds that Amaranth's positions averaged 253 days of daily trade and were in some cases 100 times the daily trade.

The firm thus faced immense liquidity risk, which explains the loss beyond that predicted by VaR. While the loss estimated by VaR was huge enough to necessitate caution, the firm's managers did not heed this as they remained leveraged at around 523 times.

In aggregate Chincarini [27] distils 3 lessons from the Amaranth failure: liquidity risk is real and should be accounted for by both exchanges and firms, regulators must push for a more meaningful measure of liquidity as VaR is inadequate and risk controls must be in place and enforced at all times.

### 3.4 Model Accuracy

Both the LTCM and the Amaranth disasters clearly highlight the need for greater model accuracy in risk measures and modelling. It seems evident that currently used, classical risk models lack some important component of market-related loss, which makes them underestimate true loss.

Risk models, like the Black-Scholes-Merton model, VaR and many others, based on classical market theory are all underpinned by the assumption of no frictions. They assume that all traders are price takers, can trade an unlimited amount immediately at zero cost or risk and face no trade restrictions [65]. In this environment the law of one price will always hold and it is fairly easy to build risk models based on normality and the weak law of large numbers.

The reality, however, is that these assumptions are extremely poor approximations of the true market price formation process discussed in Section 2.1. As a consequence, many of these models fail, particularly in times of market crises.

A few papers have tried to quantify the magnitude of this underestimation. Duffie *et al* [40] for example examine the effect of Bid-Ask spreads on the commonly used 99% VaR, Expected Tail Loss (ETL) and Default Probability (DP) of a specific portfolio. It seems to be the first, rigorous mathematical test of the effects of liquidity risk on risk measures.

Duffie *et al* [40] begin by assuming that a firm, facing capital adequacy restraints, holds 3 assets: a liquid asset  $S_1$ , an illiquid asset  $S_2$  and cash. Both  $S_1$  and  $S_2$  follow independent Geometric Brownian Motions (GBMs) through time, both of which represent the mid-price of these assets. They also model each asset's relative mid-to-bid price by  $X_1$  and  $X_2$  so that the bid price for each asset is  $S_i(1 - X_i)$ .

Next they model the firm's liquidation behaviour through time. They assume that the firm folds  $\alpha_{i,t}$  units of each asset at time  $t$  so that the initial firm asset value is given by  $A_0$ . Assuming liabilities then of  $L_0$ , the firm's initial capital is  $K_0 = A_0 - L_0$ . The firm trades an amount of  $\lambda_{i,t}$  each time period to ensure that the capital adequacy ratio of  $K_t \geq c_r A_t$ , for some fixed ratio  $c_r$  of initial capital, is maintained. The proceeds of each trade are set off against the firm's liabilities which grow by  $r$  each period. The model details are shown below.

Let  $S_{0,t}$  be a deterministic cash asset,  $S_{1,t}$  be a stochastic liquid asset and

$S_{2,t}$  a stochastic illiquid asset such that their value at time  $t$  is given by:

$$S_{0,t} = S_{0,0}e^{rt} \quad (3.1)$$

$$S_{1,t} = S_{1,0}e^{(\mu_1 t + \sigma_1 B_{1,t})} \quad (3.2)$$

$$S_{2,t} = S_{2,0}e^{(\mu_2 t + \sigma_2(\rho B_{1,t} + \sqrt{1-\rho^2} B_{2,t}))} \quad (3.3)$$

where  $r$  is a constant rate of risk-free lending,  $\mu_1$  and  $\mu_2$  are the expected return for each asset,  $\rho$  is the instantaneous correlation between the assets and  $B_1, B_2, \dots$  are independent standard Brownian motions. Let now  $X_{i,t}$  be the relative mid-to-bid spread of asset  $i$  at time  $t$  then the bid price of asset  $i$  is  $S_{i,t}(1 - X_{i,t})$ , Duffie *et al* assumes that:

$$X_{1,t} = X_{1,0}e^{(\gamma_1(\rho_1 B_{1,t} + \sqrt{1-\rho_1^2} B_{3,t}) - \frac{1}{2}\gamma_1^2 t)} \quad (3.4)$$

$$X_{2,t} = X_{2,0}e^{(\gamma_2(\rho_2(\rho B_{1,t} + \sqrt{1-\rho^2} B_{2,t}) + \sqrt{1-\rho_2^2} B_{4,t}) - \frac{1}{2}\gamma_2^2 t)} \quad (3.5)$$

where  $\gamma_i$  denotes the relative bid-ask spread and  $\rho_i$  the correlation of the mid-price increment and the change in the spread of asset  $i$ .

Now given this, suppose that the firm begins with  $\alpha_{i,0}$  units of each asset so that the firm value at time  $t_0$  is  $A_0$ :

$$A_0 = \alpha_{0,0}S_{0,0} + \alpha_{1,0}S_{1,0} + \alpha_{2,0}S_{2,0} \quad (3.6)$$

Furthermore, as noted, the firm has liabilities of  $L_0$  so that its initial capital is  $K_0 = A_0 - L_0$ . Based on the capital adequacy condition this must be maintained so that:

$$K_t = A_t - L_t \geq c_r A_t \quad (3.7)$$

Now  $\lambda_{i,t}$  denotes the units traded of the  $i$ th asset at the end of each period  $t$  so that  $\alpha_{i,t+1} = \alpha_{i,t} - \lambda_{i,t}$ . If the firm uses the proceeds from each sale to furnish its liabilities then the amount of liabilities after each trade must be

$$L_{t+1} = e^r(L_t - \lambda_{0,t}S_{0,t} - \lambda_{1,t}(1 - X_{1,t})S_{1,t} - \lambda_{2,t}(1 - X_{2,t})S_{2,t}) \quad (3.8)$$

Duffie *et al* assumes that the firm liquidates cash first then the liquid asset and lastly the illiquid asset, thus  $\lambda_{i,t}$  is determined according to the liquidation needs at time  $t$ .

a. VaR				
Spread (liquid/illiquid asset)	No Spread	0.1%/0.5%	0.2%/1%	0.5%/2.5%
Constant Spreads	6.204	6.407	6.614	7.344
Variable Spreads, $\rho_1 = \rho_2 = 0$	6.204	6.398	6.595	7.380
Variable Spreads, $\rho_1 = \rho_2 = -0.5$	6.204	6.434	6.672	7.659
Variable Spreads, $\rho_1 = \rho_2 = -0.8$	6.204	6.459	6.736	7.850
b. ETL				
Spread (liquid/illiquid asset)	No Spread	0.1%/0.5%	0.2%/1%	0.5%/2.5%
Constant Spreads	6.635	6.825	7.030	7.740
Variable Spreads, $\rho_1 = \rho_2 = 0$	6.635	6.827	7.036	7.796
Variable Spreads, $\rho_1 = \rho_2 = -0.5$	6.635	6.868	7.125	8.087
Variable Spreads, $\rho_1 = \rho_2 = -0.8$	6.635	6.895	7.186	8.273
c. Insolvency Probability (in%)				
Spread (liquid/illiquid asset)	No Spread	0.1%/0.5%	0.2%/1%	0.5%/2.5%
Constant Spreads	0	0	0	0
Variable Spreads, $\rho_1 = \rho_2 = 0$	0	0	0	0
Variable Spreads, $\rho_1 = \rho_2 = -0.5$	0	0	0	0.090
Variable Spreads, $\rho_1 = \rho_2 = -0.8$	0	0	0	0.020

Table 3.1: Duffie Test Results  
[40]

Now given this framework, Duffie *et al* [40] define their risk measures on the distribution of capital losses and gains over the entire period of trade. Thus if trade starts at time  $t_0$  and continues to  $t = 10$  then the risk measures capture the distribution of  $K_{10} - K_0$  over the period.

Starting with arbitrary base level values for the parameter set they conduct 25 000, 10-day simulations to arrive at a distribution of  $K_1 - K_0$ . They then study cases based on assumptions of the mid-to-bid spread: 1. constant spread, 2. random spreads that are uncorrelated (governed by  $\rho_i$ ) with returns, 3. random spreads that are negatively correlated with returns and 4. random spreads with positive correlations. The results of testing the risk measures for these different assumptions are presented in Table 3.1.

As shown Duffie *et al* find that the higher the initial spread and thus the higher the illiquidity, the greater the Value-at-Risk, ETL and Insolvency Probability. Clearly illiquidity, as proxied by the Bid-Ask spread, is a risk. This becomes more marked when [40] allow for non-normality of mid-prices and model returns with a jump diffusion process. To the extent then that returns become more non-normal during times of stress, the effects of illiquidity also become more onerous.

In general Duffie *et al* conclude that risk models seem insensitive to the degree of illiquidity but do respond to it as an increase in risk. While they acknowledge that price impacts are ignored in their study, they argue that the effect of liquidity on market risk must be dramatic as the spread (which is the only aspect of illiquidity they consider) only captures a small component

of total liquidity risk and yet negatively impacts the risk measures.

In a related study, Çetin *et al* [24] derive an alternate Fundamental Theorem of Finance and a new framework for arbitrage pricing in the face of liquidity-related costs. Using an extremely rigorous, mathematical foundation they show that in an economy where information flow is modelled by a filtered probability space that informs a single trader, not all instruments can be perfectly hedged. In this case even if the trader is a price taker with respect to an external stochastic supply curve (modelled by a GBM), the trader's hedging costs escalate to such an extent that hedging is impossible.

Importantly Çetin *et al* show that in such an economy not all self-financing trading strategies are unique and the market can now only be approximately complete. This means that contingent claims can only be hedged eventually, not exactly at their expiry. They even derive alternate Black-Scholes option valuations to deal with illiquidity. The implication that standard Black-Scholes fails when there is liquidity risk is significant.

Even in the context of portfolio optimization, liquidity risk and costs have important impacts. Lo *et al* [64] show that incorporating liquidity risk into a Markowitz efficient frontier can dramatically alter the efficient set of portfolios from which a potential investor would wish to choose. The potential for significant loss is thus severe.

Models which thus ignore liquidity risks and assume frictionless markets are both inaccurate and potentially costly. This is especially true for widely-used models. In such models errors can be compounded across the market and lie dormant until a fatal blow-up event reveals their inadequacies. This is the case with both LTCM and Amaranth where a single measure, VaR was heavily relied upon.

### 3.5 Problems with Value-at-Risk

*“If we ask the question: ‘Can we be 98% confident that no more than the Value at Risk number would be lost in liquidating the position?’ The answer must be ‘No’. To see why, consider what this VAR measure implies ... The following sequence of events is implied: at time  $t$  it is decided to liquidate the position; during the next 24 hours nothing is done ... after 24 hours of inaction the position is liquidated at prices which are drawn from a distribution that is unaffected by the process of liquidation.” [67]*

The widespread faith in Value-at-Risk and the risks inherent therein, necessitates a more detailed investigation into its workings and whether liquidity is adequately treated in VaR.

Although the goal of this thesis is not to undertake a wide testing of VaR models, its prominence and the fact that many of the approaches to liquidity risk build on the VaR technique make it necessary to be familiar with the significant VaR methodologies.

Initially developed and marketed by JP Morgan's *RiskMetrics*<sup>TM</sup> group in 1994, Value-at-Risk became common-place in financial institutions following the Basel Accord of 1988. This accord set general standards among G10 central banks for the capital requirements of commercial banks and explicitly allowed them to measure their capital needs with the use of a statistical model like VaR [20].

Ostensibly, the reason for the support of VaR by the Basel Task Team was that the risk measure is relatively easy to calculate, communicable and easy for stakeholders to understand. Indeed the basic concepts that underlie VaR and how these relate to market risk are surprisingly elegant.

VaR is most widely defined as that smallest critical value of a return distribution such that the probability of finding a smaller return is no more than some pre-specified probability. The exact mathematical definition for some probability space is given below.

Let  $\lambda$  be some pre-specified probability level and  $V$  a random variable in some probability space with measure  $\mathbb{P}$ .

Then the Value-at-Risk of  $V$  given  $\lambda$ ,  $VaR(V, \lambda)$  is:

$$VaR(V, \lambda) = \inf\{\alpha : \mathbb{P}(V \leq -\alpha) \leq \lambda\} \quad (3.9)$$

Now it is well-known that for any random variable  $V$  in a probability space with measure  $\mathbb{P}$ , the probability that it lies less than some real value  $\alpha$  is given by its cumulative probability density function,  $F_V$ . Thus one can arrive at the simplification:

$$\begin{aligned} VaR(V, \lambda) &= \inf\{\alpha : \mathbb{P}(V \leq -\alpha) \leq \lambda\} \\ &= \inf\{\alpha : F_V(-\alpha) \leq \lambda\} \\ &= F_V^{-1}(\lambda) \end{aligned} \quad (3.10)$$

Although the true cumulative density of the random variable is hardly ever known precisely, it can be modelled by an empirical distribution with random sampling or with a pre-specified distribution that fits the properties of the data. In this way the search for VaR can be translated into the parlance of statistical distributions and reduced to finding a density for  $V$ .

In finance  $V$  is notably the return or profit/loss on an asset. Thus the problem of evaluating VaR becomes one of adequately modelling the return distribution of an asset. This task is more complex than it seems and in reaching for a sensible model, a variety of different VaR calculation techniques have been proposed, namely: 1. Parametric modelling, 2. Kernel Estimators, 3. Monte Carlo simulation, 4. Historical Simulations and 5. Extreme-Value theory adjustments.

### 3.5.1 Parametric Modelling

The first VaR models were based on a pre-specified parametric model of the return distribution. Notably the earliest forms assumed that returns followed a multi-variate normal distribution with constant mean and variance.

In this setting the VaR of any portfolio of  $n$  risky assets reduces to:

$$VaR(\lambda, V) = -\alpha \sqrt{(W' \Sigma W)} \quad (3.11)$$

$W \rightarrow$  is the vector of portfolio weights for the  $n$  risky assets  
 $\alpha \rightarrow$  is the  $\lambda$  critical value for the assumed normal distribution  
 $\Sigma \rightarrow$  denotes the variance-covariance matrix of asset returns

In the case of a single asset, the portfolio VaR at time  $t$  becomes:  $VaR(\lambda, V)_t = V_t e^{(\mu_t - \alpha \sigma_t)}$ .

Initial attempts at making the estimation of portfolio VaR more accurate centred on changing the estimation procedure for  $\Sigma$  with the use of GARCH and EWMA models. This was done in the hopes that more accurate and dynamic estimators of volatility would better account for the empirical return distribution's skewness and kurtosis and thus make VaR more viable.

EWMA or GARCH regimes calculate VaR by updating the parametric equation with time-varying volatilities and correlations. In this way the volatility



$\sigma_{i,T}$  of the  $i$ th asset in the portfolio at time  $T$  and its correlation to the  $j$ th asset,  $\rho_{i,j,T}$ , is modelled by updating the estimator to the VaR each time VaR is calculated.

For the EWMA model, denoted here by E,  $\sigma$  and  $\rho$  are updated as follows:

$$E[\hat{\sigma}_{i,T}^2] = (1 - \lambda) \sum_{t=1}^m \lambda^{t-1} R_{i,T-t}^2 + \lambda^m \sigma_{T-m}^2 \quad (3.12)$$

$$E[\hat{\rho}_{i,j,T}] = \frac{\sum_{t=T-m}^{T-1} \lambda^{T-t-1} (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)}{\left[ \left( \sum_{t=T-m}^{T-1} \lambda^{T-t-1} (R_{i,t} - \bar{R}_i)^2 \right) \left( \sum_{t=T-m}^{T-1} \lambda^{T-t-1} (R_{j,t} - \bar{R}_j)^2 \right) \right]^{1/2}} \quad (3.13)$$

Here  $\lambda \in [0, 1]$  is a constant that determines how responsive the volatility estimator is to the most recent observation,  $R_{i,t}$ .  $R_{i,t}$  denotes the  $i$ th asset's return on day  $t$  from the end of day  $t-1$  to the end of day  $t$ , with mean  $\bar{R}_i$ .  $m$  denotes the number of days' of past observations used in the estimate [73].

Under a GARCH regime, the moving estimate of  $\sigma$  changes as shown below and a moving  $m$ -day correlation structure is used to estimate  $\Sigma$  completely [53].

$$\text{GARCH}(1,1): \sigma_t^2 = \gamma V_L + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (3.14)$$

$V_L$  denotes the long-run variance weighted by a constant  $\gamma$  such that  $\gamma + \alpha + \beta = 1$

Research into these models has shown that they perform poorly in predicting actual large scale loss and that the differences between EWMA and GARCH are minimal. Bredin and Hyde [5] show that EWMA models tend to be more conservative than GARCH but that GARCH is slightly more accurate in predicting loss. Indeed Danielsson *et al* [32] find that EWMA better reflects

the clustering of volatility around events and thus responds more quickly to new events.

The benefits of the GARCH and EWMA models lie in their ability to provide continually updating volatility estimates that place greater weight on more recent data and allows for more speedy adjustment to new information.

The drawbacks are, however, that the methods tend to perform worse when anomalous events occur which are not accounted for in the sample data set [32]. The models are constrained by the information contained in the data set and thus under-estimate absolute probable loss in that they do not account for extreme events. This is especially true of exotic instruments where estimation error is higher. They also fail to completely account for the skewness of the return distribution and provide for more volatile VaR estimates as the volatility input is updated more regularly [32].

In order to account for this, other models which use alternate distributional forms like the Student-t distribution have been used [32]. These parametric forms have been proven to improve accuracy [10]. However they still tend to under-estimate risk and are prone to estimation error.

### **3.5.2 Historical Simulation**

In an attempt to bypass the estimation error inherent in parametric models, Historical Simulation was introduced. This methodology is easy to implement and requires few assumptions. It is based on the idea that past returns are indicative of likely future returns.

With historical simulation, a period of returns is randomly selected and used to form an empirical distribution of returns. VaR is then computed from this distribution with the use of a percentile function.

Generally the methodology has more drawbacks than benefits. Firstly a sampling problem exists if an asset has too short a history to form a decent distribution. Results would be biased in this case. Problems also occur if the time period from which the data sample is drawn contains too few or too many extreme events - in this case risk would be under or over-estimated.

In addition, the historical distribution is always discrete with events less concentrated than under a parametric model. Using a percentile function in this case can lead to very biased and poor out-of-sample estimates as

data points seem to be “missing” [32]. VaR estimates thus tend to remain unchanged for a long period of time.

VaR estimates are also heavily dependant on the window chosen for the data [34]. Generally longer sampling windows can lead to more stable estimates but if the data upon which the historical simulation is based contains a high value, low probability event then all VaR estimates based on it will be biased upwards [52]. Historical simulation provides no way of controlling for normal events while accounting for extreme market moves.

### 3.5.3 Kernel Estimators

In order to correct for the data discreteness problem posed by Historical Simulation, researchers have proposed the use of a Kernel Estimator to smooth the empirical return distribution with a pre-specified functional form. In this way the benefits of both historical simulation and parametric distributions can be combined.

A Kernel Estimator is simply a non-parametric methodology for estimating the density of a random variable. It provides a technique whereby the spaces between data points in an empirical distribution can be interpolated with a kernel and a bandwidth factor [52]. The Kernel estimator for any cumulative density can be derived easily by following the argument below [52].

Let  $F_X(x)$  be the cumulative density function of a variate  $X$  then  $f(x) = F'_X(x)$  is the first derivative and probability density function of  $F_X$

Now the empirical estimator of  $F_X$  based on some data set  $\{x_1, x_2, \dots, x_n\}$  of  $n$  elements can be written as:

$$\hat{F}_X(x) = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}} \quad (3.15)$$

where  $1_{X_i}$  is an indicator function such that

$$1_{\{X_i \leq x\}} = \begin{cases} 0 & \text{if } X_i > x \\ 1 & \text{if } X_i \leq x \end{cases}$$

Now for small  $h$  a non-parametric estimator of  $f$  can be derived from first-

principles as:

$$\begin{aligned} f(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &\approx \lim_{h \rightarrow 0} \frac{F(x + \frac{h}{2}) - F(x - \frac{h}{2})}{h} \end{aligned} \quad (3.16)$$

which by replacing  $F$  with its estimator,  $\hat{F}_X(x)$ , leads to

$$\begin{aligned} \hat{f}(x) &= \frac{1}{nh} \sum_{i=1}^n 1_{\{x - \frac{h}{2} \leq X_i \leq x + \frac{h}{2}\}} \\ &= \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \end{aligned} \quad (3.17)$$

where  $K(u) = 1_{\{|u| \leq \frac{1}{2}\}}$  [52].

The above is the general form of the kernel estimator with function  $K$ , the kernel, and bandwidth  $h \geq 0$ . Generally there are a number of forms which  $K$  can assume but the form has little effect on the accuracy of the estimator [52]. There are also a number of different ways in which to estimate  $h$ .

The outcome of the smoothing is depicted in Figure 3.2 below where the solid line represents the smoothed Kernel distribution. Clearly the modelling technique allows for significantly better modelling of the tails of distributions than provided by Historical simulation or simple parametric forms [32].

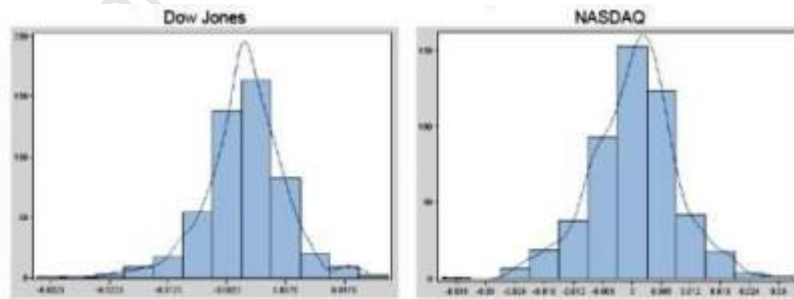


Figure 3.2: Smoothing with Kernel Estimators [52].

As shown, however, the fit of the Kernel estimator seems to improve at the tails of the distribution but performs poorly in the interior. Kernel estimators add data richness to the distribution tail without altering its

form. It is for this reason that Huang [52] propose a model that applies Kernel estimation only to the tails of distributions as opposed to its entirety. This has the added benefit that the tail is modelled accurately without changing the interior of the distribution.

### 3.5.4 Monte Carlo Estimation

VaR has also been calculated by simulating returns from some return-generating process and applying the VaR methodology to the resultant simulated distribution of returns.

Generally the methodology is based on the assumption that returns follow a geometric Brownian motion and are log-normal. Jump-diffusion processes and many other processes have, however, also been used to make the estimation of tail behaviour more accurate.

Monte Carlo simulations can be time-consuming especially for a large portfolio of complex instruments where computing time can be large. However, in many cases, the profit/loss on an instrument (like a Barrier option) can only be known with the help of a simulation. In these cases Monte Carlo simulation is the only way to compute a realistic VaR number.

### 3.5.5 Extreme-Value Theory Adjustments

Although financial asset returns are most commonly modelled with the use of a normal or log-normal distribution, it is a well-known fact that returns are actually non-normal, skewed and fat-tailed [45]. As a consequence VaR estimates based on normal, thin-tailed distributions tend to severely underestimate true maximal loss.

Typically returns, especially if viewed at high frequencies, exhibit significant volatility clustering with extreme events, which are not easily captured by Value-at-Risk based on the standard distributions [32]. Actual return distributions usually cannot be described by their moments as many higher-order moments are infinite. They need to be described by quantiles [45].

In order to correct for this, researchers have developed Extreme-Value theory (EVT) measures and adjustments to help VaR more accurately account for extreme events. In this way, large-scale low probability events like

widespread market crises are not underestimated and users of VaR get a better idea of potential maximum loss.

Extreme value theory is based on the notion that values of a random variable above some specified threshold follow a generalized Pareto distribution provided that certain conditions are met. This provides an efficient way to obtain information on the tails of a distribution from the empirical distribution of the variate.

The underling basics of EVT are discussed below. It in no way encompasses all the EVT models but merely highlights the basic characteristics of most of these models.

If a random variable  $X$  follows a generalized Pareto distribution then

$$F_X(x) = \mathbb{P}(X \leq x) = G_X^{\alpha, \beta}(x) = \begin{cases} 1 - (1 + \frac{x}{\alpha\beta})^{-\alpha} & \text{if } \alpha < \infty \\ e^{-\frac{x}{\beta}} & \text{if } \alpha = \infty \end{cases}$$

where  $\alpha \in (0, \infty]$  is the tail index of the distribution [45].

Now under mild conditions on the distribution function of  $X$ , one can find a positive  $\beta(u)$  such that

$$\lim_{u \rightarrow \infty} \frac{1 - F(u + \beta(u)x)}{1 - F(u)} = (1 + \frac{x}{\alpha})^{-\alpha} \quad (3.18)$$

where  $\lim_{u \rightarrow \infty} \beta(u) = \frac{1}{\alpha}$ . If moreover  $U$  is high enough and one puts  $Y = (X - U)^+$ , the excesses above a threshold, then the distribution of  $Y$  is given by

$$\mathbb{P}(X < U + y \mid X > u) = \frac{F_Y(U + y) - F_Y(U)}{1 - F_Y(u)}, \quad y > 0 \quad (3.19)$$

and this follows a generalized Pareto distribution  $G_Y^{\alpha, \beta}(y)$  [52].

The above result can be used to derive a model of the entire distribution of some random variable  $X$  such that the interior values are derived from the empirical distribution and the tails from a fitted Pareto distribution. The details of this process, called the Peaks over Thresholds method is shown below [45].

Given a random variable  $X$  with unknown distribution  $F_X(x)$ , the peaks over thresholds method can be applied to model  $F_X(x)$  more accurately.

Since  $F_X(x) = F_X(u) + F_X(x \mid X > u)(1 - F_X(u))$  if  $x > u$  for any  $u$ , then one can estimate  $F_Y$  as follows:

1. Fix a threshold level  $u$
2. Select  $Y_1, Y_2, \dots, Y_k$ , the positive values of  $(X_1 - u)^+, (X_2 - u)^+, \dots, (X_n - u)^+$  for a data sample of size  $n$  for  $X$
3. By the previous result then  $Y_i \sim G_Y^{\alpha, \beta}$
4. If one now fits  $G$  and estimates  $\alpha$  and  $\beta$  by Maximum Likelihood then the peaks over thresholds estimate  $F_X(x)$  is:

$$F_X^{\hat{}}(x) = \begin{cases} \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}} & \text{for } x \leq u \\ F_X^{\hat{}}(u) + (1 - F_X^{\hat{}}(u))G_Y^{\alpha, \beta}(x - u) & \text{else} \end{cases}$$

The only problem with the above distribution estimator is locating a suitably high value for  $u$ . A value for  $u$  can be found by using the Hill estimator or by plotting the empirical shortfall for different  $u$  and selecting the smallest value such that the shortfall does not change.

The empirical shortfall of a random variable  $X$  is

$$s(u) = \frac{1}{k(u)} \sum_i (X_i - u)^+ \quad (3.20)$$

where  $k(u)$  is the number of observations that exceed  $u$  from the empirical estimation sample.

By plotting  $s(u)$  against  $u$  one can determine the smallest  $u$  which seems reasonable.

Alternatively one can use the Hill Estimator to find  $u$ : If  $X_{1,n} \geq X_{2,n} \geq \dots X_{n,n}$  are the order statistics of  $X$ .

Then if one fixes a threshold  $u$  with  $k(u)$  as above then the Hill estimator,  $\hat{\alpha}$ , of  $\alpha$  is

$$\frac{1}{\hat{\alpha}} = \frac{1}{k(u)} \sum_{j=1}^k \log\left(\frac{X_{j,n}}{X_{k,n}}\right) \quad (3.21)$$

then the best choice of  $u$  is that value that yields a constant and stable estimate of  $\alpha$ .

While the above does not encompass all the EVT models, it touches most of their characteristics. Generally EVT models provide more accurate and stable VaR estimates than other models. Not only are the predictions more accurate but when the VaR estimate concerns itself with low probability

events, the EVT estimate performs considerably better [32]. EVT thus does what it sets out to do, capture extreme events. These benefits, however, comes at the price of ease of application.

Similar to historical simulation, EVT only captures extreme events if the initial data set includes rare events. A lengthy data set is better. Indeed Danielsson *et al* [32] advocate the use of at least 7 years of daily data. Moreover the estimation procedure is fairly complex and difficult to explain to stakeholders.

EVT VaR generally outperforms standard methods as it provides more reliable estimates of true market loss. Danielsson *et al* [32] contend that this is true because the volatility clustering and distributional assumptions used to make classical VaR more accurate becomes of little use during extreme market events as returns become more random.

### 3.5.6 Comparing the Methodologies

The natural question which arises from discussing the array of VaR methodologies is how the different estimation procedures compare. Fortunately this is a topic which a number of papers have investigated with the assistance of back-testing.

The majority of VaR models are tested by estimating the model on a set of data and then applying it to a separate data set and evaluating the predicted outcomes versus the actual outcomes. This is back-testing. In VaR, back-testing is achieved by counting the number of times a portfolio's actual market loss (over a post-fit data set) exceeds that predicted by VaR. If the VaR estimate is breached too many times relative to the stated probability then the model is poor and underestimates risk. If the VaR losses are hardly ever breached then the model is also poor as it is too onerous. An accurate  $x\%$  VaR should only be exceeded by the realized loss  $1 - x\%$  of the time. This is hardly ever the case however.

VaR is extremely model dependant and different methodologies provide for extremely divergent estimates of loss. The measure is also highly data dependant and different data sets can impact on it heavily. VaR is not robust. Indeed Beder [15] has found, through a survey of financial institutions using VaR, that there are radically different methodologies of estimation. Moreover there are also great discrepancies in results for different methods applied to the same portfolio. This is true even for a simple relatively risk-



less portfolio holding only US Treasury Bills and the discrepancies widen in comparing the results of different methodologies.

In comparing Monte Carlo, historical simulation and variance-covariance VaR measures, De Raaji *et al* [34] find extreme variability across methodologies. These differences were in the order of 25% to 59% but sometimes approached 200%. They find that the variance-covariance and historical simulation techniques are weaker at lower probability levels, involving the tails than at higher levels. This is indicative of the poor accounting for extremes inherent to these models. Interestingly a Monte Carlo simulation based on mixed distributions with a fat-tail adjustment produced the most accurate results. The paper concludes that it would be highly misleading to compare VaR numbers from 2 different entities without regard to their methods of calculation.

Using a similar procedure but this time comparing models to an EVT sampling procedure, Danielsson *et al* [32] find that a VaR based on a mixed distribution with tails modelled by a Pareto distribution and the interior modelled by a historical simulation, performs better than standard variance-covariance VaR at high probabilities and seems to estimate risk fairly.

Huang [52] conduct a wider study comparing variance-covariance, historical simulation, Monte Carlo simulation and a kernel estimator used on tail distributions. They find that their revised Kernel methodology with only tails modelled by a kernel estimator performs remarkably better than the other models. Unfortunately the result is not compared to the EVT adjustment for tails thus one cannot infer which of the 2 methods are more accurate. The Kernel estimator does, however, require far less data and computation than the EVT procedure.

In spite of the onerous data requirements, EVT models with a Pareto distribution for the tails offer more useful estimates than other methods. This is true for a number of reasons. Firstly by concentrating on the tails, such EVT tail modelling overcomes the problem inherent to historical simulation of an upwardly biased interior distribution. In this way both extreme and normal events can be accommodated. Moreover unlike the discrete historical simulation distributions, EVT tail-based modelling tends to provide more stable, low variance estimates of VaR which are based on the firmer statistical foundations of extreme value theory. These estimates also provide more prudent and reasonable estimates of worst-case loss [45].

In aggregate although EVT models are laudable, even EVT VaR has a high probability of underestimating true risk, particularly in large organisations.

VaR is not a coherent measure and, more importantly, in any of the above forms, it ignores market-related liquidity risk.

### 3.5.7 VaR and Liquidity-Related Risk

As implied in the opening quote of this chapter and as stated by Hisata *et al* [49], “conventional VaR assumes immediacy, no price impacts and no friction”. VaR is meant to be a “statistical measure that allows one to capture the amount of losses likely to be recorded from a market movement ...” [44]. By ignoring price impacts, transaction costs and other frictions, VaR clearly underestimates the losses likely to be faced by a market participant. It underestimates true risk.

Conventional VaR (based as it is on a pre-specified distribution) assumes that market losses follow a specific distribution and that these losses are realized by liquidating at the mid-price or close-price of the day. It also assumes that a participant’s entire portfolio is sold in one large block with no price impact at the date of trade.

As argued earlier none of these assumptions hold true. In one particular day a trader may face any number of prices at which to liquidate. Moreover as traders usually stagger their trades, the sale of one large block usually occurs in many different pieces at many different realized prices. VaR assumes that the portfolio is frozen over the holding period [56] and thus ignores all the losses borne of trading: the opportunity costs of waiting to trade and the price-impact cost of actually trading. In fact VaR even ignores brokerage and other fixed costs. It is for this reason that the validity of VaR is questioned, particularly in large organizations with many different trading desks, positions and instruments [17].

VaR does offer a simple ad hoc technique for accounting for liquidity risk in the form of adjusting the holding period over which returns are calculated. By changing the returns used in the return distribution from say a 1-day return to a 1-month return, liquidity is somewhat accounted. This is reflected in the fact that the 1 month VaR is higher than the 1 day VaR. There are problems with this methodology, however.

Firstly such a holding period adjustment tends to be arbitrary and still ignores many liquidity-related costs [20]. In addition such a “scaling adjustment” worsens with the jump intensity of returns and the confidence level [33]. The adjustment is only relevant if logarithmic returns are identical,

independently distributed normal variates [36]. The majority of papers that mention the scaling adjustment disapprove of it.

Bangia *et al* [14] for instance heavily criticise the methodology and argue that given the rising expansion of investments into illiquid regions, the need for a liquidity risk adjustment to VaR becomes increasingly necessary. François-Heude *et al* [44] concur and point out that the adjustment is highly subjective. In addition Shamroukh [76] argues that the scaling adjustment precludes the existence of securities in a portfolio with different times to maturity. The technique is an inadequate adjustment for VaR.

The pressing importance of liquidity in market-related risk, coupled - as we have seen with Amaranth and LTCM where classical VaR was widely applied to no effect - with the inaccuracies which can result from blind belief in VaR, makes it necessary to find an alternate measure of total risk.

The widespread popularity and intuitive sense that makes VaR commonplace means that such a measure should take its roots from the standard VaR framework but should include accuracy adjustments for liquidity risk.

However, merely because a methodology is popular and widely used does not imply that it is the best measure for the job at hand. In this respect liquidity-adjusted VaR, may not be the most accurate liquidity risk measure in existence and the full gamut of available measures should be tested and explored.

### 3.6 The Elements of a Good Liquidity Measure and Model

The search for a “best-case” liquidity-risk measure necessitates a clear and focused discussion around what constitutes “best-case” and what defines a good model or measure of liquidity risk. Although in modelling, perfect accuracy is always the ideal, this is rarely achieved and a number of other criteria must be considered to find the ideal model.

In this regard, Loebnitz [65] argues that two of the most important criteria characterising a good measure are “internal consistency” and “measurability”.

Internal consistency refers to the inter-relationships of the model to other aspects of theory and known, stylized facts. It occurs when the results

of a model does not contradict its endogenous details, assumptions and implications. A good, internally consistent model should appear as a unified body of work with scope for additions but not substantive changes.

In doing this, however, a model should also remain measurable in that it conveys useful and interpretable information upon which users can base objective decisions. A measurable model is thus accurate. Moreover the extent of the accuracy can be tested and assessed.

A good measure should not only be descriptive but should also accurately indicate the cost of not heeding the information it conveys. Measures like Value-at-risk and Expected-Tail-loss are particularly good at this as they provide bottom-line, value information which the user can quickly interpret and understand. Models like this allow a user to respond quickly to new information so as to limit their loss. They also allow the user to judge when they are incorrect - for example if the estimated loss is too high or too low for the accepted level of risk.

The risk, of course, in defining a model in this way is that it may be too simplified. As discussed, while VaR is extremely measurable and succinct, it also underestimates true market-related risk. A measure must balance the need to be interpretable and easy to implement and the need for accuracy. In particular a liquidity risk measure should capture the multiple facets of liquidity and yet remain compact. This is made somewhat difficult by the breadth of characteristics that define liquidity.

Fernandez [42] argues that while a good liquidity measure must capture many disparate issues, this task is complicated by the sheer volume of data needed to provide accuracy and meaning to such a broad issue. Data constraints are a problem with liquidity measurement as generally high frequency trade data is required, specifically for the modelling of the price impacts which define it [28]. In this way many liquidity models may be extremely accurate (in that they use all available data) but intractable as the data cannot be sourced.

Many papers have thus pointed out that finding a single useful measure of liquidity could be impossible [28]. The favoured recourse is to use a wide range of measures, each of which reflects different aspects of liquidity [8]. In this way all aspects can be monitored and traded off one another to make the most objective decisions. The dilemma, however, lies in finding a method with which to differentiate between models so as to isolate that subset of most optimal measures.

The task of differentiation is made simpler by clearly outlining the ideal

model. From the above, it can be concluded that an ideal model of liquidity risk should possess the following qualities:

1. It should be measurable and should directly capture the cost of liquidating in a market at a specific time
2. It should represent the market consensus of liquidity over time and across asset classes [43] and should be practicable in all asset classes
3. It should be internally consistent with certain stylized facts and theory regarding market liquidity risk
4. It should be accurate in that the information it presents should be a fair reflection of reality

Items 2 and 3 above allude to the discussion in Chapter 2 as to what constitutes liquidity risk. The need to directly capture the costs of liquidation and certain stylized facts pertaining to market liquidity implies that models and measures of liquidity risk should, at the maximal level, incorporate some of the following:

1. Explicit transaction costs – while these do not add to the uncertainty relating to the liquidation value of a holding, they do reduce the expected proceeds and should be incorporated.
2. Bid-Ask Spread volatility – as stated, higher spread volatility increases the uncertainty in the expected proceeds from liquidation.
3. Absolute level of the Bid-Ask Spread – other things equal, a higher spread is suggestive of higher trading costs and greater variability in the expected value-under-liquidation.
4. Price-Liquidity risk trade-off – measures must account for the fact that the longer the time,  $\tau$ , between trades, the greater the price risk borne by the trader, while shorter  $\tau$  may induce larger price-impacts. Models thus need some form of  $\tau$ -penalty function that is increasing in the order size. Modelling this trade-off also helps model the bulk of the opportunity costs associated with trading.
5. Price-impacts – certainly models should account for price-impacts, both endogenous and exogenous and, if possible, temporary and permanent impacts. A good model would capture the stated empirical evidence that buyer-initiated impacts are larger than seller-initiated and that serial return-dependencies exist.

6. Market crises – the rising price-impacts, greater volatility and greater importance of exogenous risk during times of stress should be modelled. Models should also not ignore the fact that during times of stress, the correlation between asset returns and price-impacts becomes increasingly negative.
7. Other Aspects – the ideal model would incorporate all aspects of liquidity, particularly in times of stress in order to model relative risk behaviour. The other aspects like depth and breadth, determine the level of risk endemic to a market. The model should thus be able to be broadly descriptive as well.

As can be seen the requirements of the “ideal case” are onerous and, given the data constraints that over-shadow all liquidity-risk modelling, seem almost impossible to attain. Luckily, however, much of this difficulty can be overcome with the use of carefully-chosen liquidity measures instead of merely one model. By relying on many measures, the pressure placed on a single measure to account for all aspects of liquidity and to do so with easy-to-find data is alleviated. All of this is done while more information is conveyed.

The proposed use of many measures, of course, necessitates that each measure/model be assessed with regards to accuracy, tractability and completeness. These details and the methodologies applied to compare the models is discussed, at length, in Chapter 5.

## Chapter 4

# The Gamut of Liquidity Risk Measures and Models

The search for an accurate account of liquidity risk has gained increased momentum of late with the literature addressing the subject from a number of different perspectives. Research has largely centred on the pricing and importance of liquidity risk variables in asset returns, determining if liquidity variables have a systematic and a unique risk component and assessing to what degree different measures capture different information. All of this has culminated in the more detailed and functional L-VaR models.

Although the L-VaR models represent a promising step towards a complete and tractable model of liquidity risk, much of the literature has ignored the role which the Measures can play in hinting at liquidity risk. Research into the measures has focused more on whether they proxy for liquidity risk at all and, if they do, what components of liquidity risk they seem to capture. Very little work has been done in terms of how the Measures dovetail with the definition of liquidity risk.

The goal of this section is to present the major liquidity risk measures proposed in the literature together with a short account of the research pertinent to them. Unlike the asset pricing research, the focus is on assessing how well different measures perform in the management of liquidity risk, not on determining if they co-vary with asset returns. Much of the detailed asset pricing work is thus ignored in favour of a discussion of the measures themselves – whether there is a basis for them as a liquidity measure and what component of liquidity risk they seem to capture. The focus is on assessing how well the measures/models meet the criteria of Section 3.6 and

the definition put forth in Chapter 2.

Overall the literature has focused on the following broad groups of liquidity measures:

1. Spread-based measures
2. Trading activity-based measures
3. Price-impact measures
4. Liquidity-VaR Models

## 4.1 Spread-Based Measures

The principle spread-based measures are the: Bid-ask spread, effective spread and relative effective spread.

$$\text{Bid-Ask Spread}_t = A_t - B_t \quad (4.1)$$

$$\text{Mid-Price}_t = M_t = \frac{1}{2}(B_t + A_t) \quad (4.2)$$

$$\text{Effective Spread}_t = P_t - M_t \quad (4.3)$$

$$\text{Relative-Effective Spread}_t = \frac{1}{M_t}(P_t - M_t) \quad (4.4)$$

$B_t \rightarrow$  Bid price

$A_t \rightarrow$  Ask price

$P_t \rightarrow$  Transaction price

The motivation for using these spread measures and, in particular the Bid-Ask spread, is that they capture the costs associated with reversing a small



trade [43]. As discussed earlier in Section 2.3.1, the spread exists because of asymmetric information, inventory control costs and other consequences of market friction. It captures the cost of processing orders, the size and volatility of accumulated order flows [75] and is the most basic cost associated with trading [9]. At least spuriously, it seems to be the most obvious candidate as a measure of liquidity risk and commonly appears as the first choice liquidity-risk variable. The literature has, however, raised some misgivings with regard to its use in risk management.

Firstly the B/A spread cannot be used in an OTC market or any market where there are few bids and asks or where these are unrecorded. Moreover the cost it captures becomes proportionately smaller in times of stress when liquidity risk becomes a more significant component of market-related risk and is dominated by price impacts [79]. It is not a relevant measure in times of stress [42].

The spread is also only relevant for trades which occur at the bid and ask - trades that occur outside the spread face larger costs and those occurring within it face smaller costs. It is also made ineffective by the minimum tick sizes imposed. These make the spread discrete and limit the information it conveys [26].

In order to provide a measure with a surer footing than the spread, researchers have justified the use of the effective and relative effective spreads.

In a frictionless universe the spread would be zero as the bid and ask prices converge to one another. Thus only one price would rule at a specific time and the best proxy for it, given the convergence, would be the mid-price. The mid-price represents the ideal price of an asset; that is the price which would rule had markets been frictionless. Given this, the effective spread and relative effective spread capture the price effectively “paid” due to frictions.

Unfortunately, although widely used, the effective spread has its own flaws. Firstly the mid-price in no way measures the intrinsic value of a security, thus it cannot propose to accurately capture the cost due to liquidity [78]. As discussed earlier, liquidity costs involve price impacts, the spread and many other components.

The measure also fails during times of stress. During a particularly illiquid period, for example, the trade-price may suddenly fall as trade becomes more one-sided and sellers flood the market. If transactions then only occur sporadically  $M_t$  could still remain high from older asks, meaning that  $P_t$  could become close to  $M_t$ , thereby falsely implying that markets are liquid when they are far from it. The measure is thus prone to extreme noise

and is relevant only for short periods at a time. As shown later, this poses problems for liquidity models which derive their estimates from it.

Overall spread-based measures merely serve as a rough gauge of only one aspect of liquidity [43]. They are prone to extreme volatility both intra-day and over longer time periods (the noisier a measure, the less stable its information signal) as they are easily moved by large outlier trades [75]. The measures only represent a portion of the total likely loss in trading.

Indeed research in other markets seems to indicate that they only capture market depth <sup>1</sup> and ignore other aspects of liquidity risk [43]. This one-sidedness may seem like an additional shortcoming, however, the persistence with which spread-based measures are quoted in liquidity literature and the continued evidence that they are priced in asset returns, correlated with other measures and indicative of illiquidity ([43], [29], [9], [72]) seem to indicate that they are the best measures of depth available. All other measures seem to focus on other aspects of liquidity.

The idea that different measures capture different aspects of liquidity has support among the literature with Aitken and Comerton-Forde [4] showing just this. A case can thus be made for the continued use of spread-based measures in normal times when depth is more indicative of illiquidity, coupled with supporting price-impact measures for times of stress. The measure cannot be used in isolation to manage liquidity risk.

## 4.2 Trading-Activity-Based Measures

Chollete [29] argues that liquidity measures can be bisected into trade-based measures and order-based measures. Trade-based measures reflect “consummated liquidity” or executed order information while, order-based measures capture information on orders prior to their execution. Spread-based measures are trade-based, while trading-activity measures like Volume traded, turnover, quote size, trade size, number of quotes, order imbalances, trade frequency, etc. are order-based.

Unlike the spread-based measures there seems to be no theoretical basis for the use of trading-activity measures other than the fact that they proxy

---

<sup>1</sup>A market with depth has large volumes available at many different prices. The spread will thus be tight as the market-maker faces little costs in reversing a position to which he has committed himself. In effect the market-maker will face a lower opportunity cost in providing immediacy hence the pass-on cost to traders through the spread is lower

for market breadth [31]. They do not capture the costs attributable to liquidation but serve as an indicator of the potential for liquidating large orders. Provided the market has depth as well, a trader of a large position can be reasonably sure of a trade. If the market lacks depth, of course, such trade may occur at poor prices but at least the order can be filled.

While the literature cites a large number of trading-activity measures, the most commonly-used ones are listed below:

**Volume Traded** — usually the average of the total number of shares/bonds, etc traded over a specified period. Upper [79] argues that it is a poor proxy for liquidity as it is sensitive to the period over which it is calculated. Generally traders prefer to trade unevenly over extended time periods, thus if volume is only measured over a day, it would underestimate true breadth. They argue that one should use volume together with the B/A-spread as quite reasonably, “higher spreads seem less troubling if volumes also rose when they occurred”. Fleming [43] and many other papers also point out that the measure proxies for market volatility. It is thus a rough measure and could be high merely because new information is impacting on prices not because of higher liquidity.

**Quote Size** — the quantity explicitly bid or offered at the bid or offer price, usually an average over some period and cited differently for bids and offers [43]

**Trade Size** — the quantity actually traded at the bid or offer prices

**Turnover** — commonly used for shares but can be used for bonds. It is the volume traded in the share divided by the number of shares outstanding [58]. Lee et al [62] question the role of turnover as a proxy for liquidity. They argue that high turnover stocks are usually glamour stocks, while low turnover stocks are usually poor performers - thus turnover proxies for asset performance. Amihud and Mendelsohn [26], however, contend that in a frictionless economy investors would re-balance their portfolios all the time and turnovers would be high thus lower turnovers are indicative of market friction and illiquidity. Hu [51] examines turnover on the Tokyo Stock Exchange over 17 years and finds strong evidence that higher turnover is associated with lower expected returns.

**Trade Frequency** — the number of trades initiated and executed over a time period [43]. This measure is also strongly associated with market volatility.

**Order Imbalances** — the number of buy orders less the total sell orders over a particular period [30]. This measure conveys more information than the preceding ones as it gives an indication of the direction of price pressure. Chordia *et al* [30] indeed find strong evidence of contemporaneous association between returns and order imbalances.

Each measure has its own pitfalls and strengths. However, overall volume and share turnover seem to be the predominant measures with the others earning less significant results in asset pricing tests [29].

Keene *et al* [58] for instance undertake an analysis whereby stocks are ranked and sorted on the basis of their market capitalization, their book-to-market value ratio and liquidity variables. They form 54 portfolios by dividing stocks on the NYSE into 3 book-value/market-value trisections, 2 size bisections, 3 liquidity trisections (with liquidity proxied by volume traded, share turnover, and others) and 3 momentum bisections. They additionally compute the returns on a portfolio based on the intersection of 2 size and 2 liquidity variables, thereby creating factor-mimicking portfolios for each of liquidity, size and book/market value. By regressing the returns of the liquidity portfolios against each of the size, momentum and book/market value factor-mimicking returns they arrive at a residual term that captures, in their opinion, returns only due to liquidity risk and not to the other variables. In this way they isolate the effect of liquidity and can judge if it is priced. In aggregate they find that volume and share turnover show the most robust pricing effects.

Fleming [43] finds similarly by regressing a stock's 5-minute price change against its trading frequency, trading volume and other measures. They infer that trading volume impacts on returns more strongly than the other measures. They also find that trade volume is more correlated to other measures of liquidity than for instance trade size or quote size. From this and many other studies, one can conclude that the other measures are somewhat less indicative of liquidity risk.

Generally the trading-activity measures capture different information to the spread-based measures. This is the rationale behind the trade-based vs order-based differential noted earlier. Although the exact scope and strength of the distinction between the measures is not known and should be researched, the above findings make it clear that trade-based measures remain relevant.

### 4.3 Price-Impact Measures

Price-impact measures represent the first move in liquidity-research towards directly computing the cost associated with liquidation. They have a sound theoretical backing in that they proxy for the endogenous risk that is extensively cited in market micro-structure literature.

Possibly the first price-impact measure is Kyle’s lambda. Developed by Kyle [59], the measure is defined as the slope of the regression line that relates transaction price changes to the net trade sizes which are associated with it. The motivation behind the measure is Kyle’s argument that “spreads are an increasing function of the probability of facing an informed trader” [72]. Since market-makers cannot distinguish between informed and uninformed trade, they set prices as an increasing function of the net order imbalance. Unlike preceding measures, its computation requires high-frequency intra-day data which may be difficult to obtain. The measure cannot distinguish between liquidity-related price impacts and the effect of new information on asset price changes. Volume as a measure of activity problematically ignores these subtleties.

Related to the Kyle Lambda is the Pastor & Stambaugh measure of return-reversal. In their 2003 paper, Pastor & Stambaugh argue that liquidity must feature strongly in asset pricing returns. They point out that illiquid stocks carry additional risk over and above market risk, thus the rational investor must expect a premium for holding them as an incentive. Although they concede that liquidity has many dimensions, they introduce the measure,  $\zeta$ , to test their hypothesis regarding the temporary price impacts accompanying order flow [70].

$\zeta_{i,t}$  in 4.5 is the Pastor & Stambaugh measure of volume-related return reversal for stock  $i$  in month  $t$ , calculated as the ordinary least squares (OLS) regression coefficient of the equation.

$$r_{i,d+1,t}^e = \theta_{i,t} + \phi_{i,t}r_{i,d,t} + \zeta_{i,t}\text{sgn}(r_{i,d,t}^e)V_{i,d,t} + e_{i,d+1,t} \quad (4.5)$$

For  $d = 1, \dots, D$

$r_{i,d,t} \rightarrow$  return on the  $i$ th stock on day  $d$  in month  $t$

$r_{i,d,t}^e = r_{i,d,t} - r_{m,d,t}$ , the excess return of the stock over that of the market index  $m$

$V_{i,d,t} \rightarrow$  the volume of the  $i$ th stock on day  $d$  in month  $t$

The thinking behind the measure is that order flow, as proxied by the product of volume and signed excess return, should be accompanied by a return that is partially reversed in the future (in this case  $d+1$ ). Thus price changes accompanying large volumes reverse when liquidity is low because market-makers tend to require higher expected returns in order to accommodate traders [26]. Pastor & Stambaugh (P&S) argue that the “greater the expected return reversal for a given dollar volume”, that is to say the greater the  $\zeta$ , the lower a stock’s liquidity. They expect  $\zeta$  to be negative most of the time and larger in absolute value for stock’s which are more illiquid.

The major drawbacks of the P&S measure is that if there is asymmetric information between market-makers and traders then the volume-return reversal relationship is weakened by informed trading. In this case the volume contains new information that impacts on prices permanently and which is not reversed. Lee & Swaminathan [63] provide strong evidence that in this case the estimate of the return-reversal effect could be under-estimated by the regression specification [70].

Porter [72] also contend that other issues may at times undermine the use of dollar volume in the above regression equation. They argue that depending on the number of shares outstanding, the differences in free-float and investors trading behaviour, a given volume figure can have very different impacts on a share’s return. Thus one may estimate a similar  $\zeta$  number for different stocks, when in actual fact the state of liquidity between them is rather different. They argue that the use of Dollar Volume/Market Capitalisation more accurately controls for these differences.

Despite these problems the measure has been subjected to much investigation in the asset pricing literature. Without going into the details of their methodology, Pastor [70] and many other papers have found that their measure is priced across securities, exhibits sharp declines which co-ordinate with known market declines and exhibits commonality across stocks. Porter [72] finds the same results and notes that the P&S measure shows high correlations with other measures of price-impact.

One of these additional price-impact measures is the Amihud illiquidity ratio. According to Acharya *et al* [3], a stock is illiquid and thus has a high value for the ratio if its daily return for a give volume is high. Amihud [8] shows that the measure is positively related to the costs of selling and the price impact of a stock. The measure captures the change in an asset’s price in response to order flow. It is highly correlated to the Kyle’s lambda and has been found to be the “best available price-impact proxy constructed from [easily available] daily data” [26].

$$\text{Amihud Illiquidity Ratio}_t = \frac{1}{D_t^i} \sum_{d=1}^{D_t^i} \frac{|R_{t,d}^i|}{V_{t,d}^i} \quad (4.6)$$

$R_{t,d}^i \rightarrow$  return of the  $i$ th stock on day  $d$  in month  $t$

$V_{t,d}^i \rightarrow$  volume of stock on day  $d$  in month  $t$

$D_t^i \rightarrow$  the number of observations attached to the  $i$ th stock in month  $t$

Acharya *et al* [3] calculate the innovations associated to the measure and find that spikes in this time series correspond to well-known illiquidity events. The innovation series' correlation to the innovations of the P&S measure is also quite high. Moreover the measure is significantly priced.

The most notable objection to the Amihud illiquidity ratio is that it only approximates the price impact; it does not actually measure the cost of trading. However, the same can be said of all the other measures - at best they merely proxy for the true cost associated with illiquidity, with different measures capturing different aspects of the cost.

In particular, all of the price-impact measures seem to capture some aspect of market resiliency. As discussed earlier, resiliency refers to how easily a market absorbs shocks induced by large order flow. Price impacts attempt at measuring exactly this as they capture the cost associated with large order flows. In doing so, of course, they ignore depth and breadth.

## 4.4 Assessing the Measures

The above discussion focuses on introducing the literature's most widely-used measures in organic groupings that reflect different aspects of liquidity. Measures cited within a grouping are expected to, at least theoretically, specialise in capturing a single aspect of liquidity risk.

If measures in different groups only capture a particular aspect of liquidity that is independent of other aspects than correlations between measures which do not share a group should be low. Moreover a PCA undertaken on these measures should clearly reveal independent factors/components which have high factor loadings on measures according to their grouping. Thus if

a PCA were undertaken on say 5 measures with 3 from a specific grouping and 2 from another, the PCA is expected to reveal at least 2 factors with high loadings on at least one group. In this way the correlation analysis and the PCA should support the theoretical findings.

The empirical research, however, finds only mixed support for the theoretical differences in the measures. In some tests measures from different groupings have been found to be correlated in surprising ways and a PCA has only revealed weak support for the existence of independent sources of variation that relates neatly to the above theoretical split. Although the time series plots of the measures and their innovations show surprising co-ordination with periods of illiquidity, this only reveals that the measures similarly capture variations in general liquidity. It is silent on whether the measures hold different information.

Fleming [43] tries to discover whether certain measures of liquidity in the US Treasury Bill market capture different aspects of liquidity. Using trading volume, trading frequency, bid-ask spreads, quote sizes, a price-impact measure that is similar to Kyle's lambda and trade sizes as proxies for liquidity, Fleming firstly plots the time series of the measures and examines how well they capture declines in general liquidity as in Figure 4.1 below. He finds that largely all the measures show similar declines and form similar trends.



Source: Bloomberg.

Notes: The thin line represents the Treasury yield; the thick line represents the target rate. LTCM is Long-Term Capital Management.

Figure 4.1: Time Series plots of Volume [43].

Secondly he undertakes a correlation analysis whose results are revealing. Fleming finds that the bid-ask spread has very low correlations with the



other measures (which is understandable given that the rest are mostly trading-activity measures), except for the price impact measure with a 0.73 correlation. This is surprising given that price-impact measures are meant to capture resiliency while the spread proxies for depth. In general, however, all of the measures have high correlations with the lambda measure. This may reflect the fact that weakness in depth, breadth and resiliency in a market leads directly to higher costs of execution in the form of higher price impacts. If this is true then a positive relationship is expected between price impacts and the other measures which are increasing in illiquidity. Fleming's results largely reflect this as his price-impact measure is only negatively related to the trade size and quote size. He offers no explanation for this.

The results of a PCA undertaken on the same measures finds that 3 components explain 87% of the combined variation in the measures. He interprets the first component as capturing variation in liquidity that is negatively related to trading activity, while the second and third components capture variation that is positively related to the same. Both measures show very weak correlations to the bid-ask spread which seem to indicate that the bid-ask spread does indeed capture different information to the trading activity variables. Fleming [43] uses this, incorrectly, to conclude in favour of only using the bid-ask spread as a measure of liquidity. He argues that given its high correlation to the price-impact measure and easy accessibility, the measure is ideal. Of course, in saying this, he ignores the suggestion conveyed by his results that the spread captures a different aspect of liquidity and must be supported by other measures.

In a very similar study Chollete *et al* [29] investigate, with a correlation analysis and a PCA, whether the monthly number of trades, monthly trading volume, average trade size, the Amihud illiquidity ratio, average quoted spread, average relative spread, average effective spread, intra-day and daily return volatility contain information on different aspects of liquidity.

Chollete *et al* [29] provide a diverse set of correlation results. They find, as expected that the spread measures are all highly correlated, but exhibit low correlations with all the other measures. They do, however, display a moderate correlation to volatility. Their results show clear separations between trade-activity measures, spread measures and the price-impact measure, as represented by the Illiquidity ratio which is not significantly correlated to any other measure. Interestingly bid volume and ask volume display weak correlations to all the other measures as well. This may imply 2 things: either they capture a different aspect of liquidity or that they do not proxy well for liquidity at all. The first contention seems more reasonable given the poor results with these measures across the literature.

The factor analysis undertaken by Chollete *et al* [29] extract 3 factors, which explain 51%, 30% and 19% of the variation in the measures respectively. In order to determine if the factors convey general liquidity information they plot each factor over time. In aggregate, as with the measures, the factors do reflect periods of general illiquidity. In order to determine if different factors convey different information, Chollete *et al* [29] examine the factor loadings on each variable. They conclude that “there is differential information in the different liquidity variables”. In particular they interpret Factor 1 as a price-impact measure that captures resiliency as it loads highly on the illiquidity ratio and return volatility. Factor 2, they infer, must be related to trading activity as it loads on breadth and depth variables, while Factor 3, they call a depth measure.

Compellingly, Chollete *et al* [29] find in their asset-pricing tests that Factor 1 only attracts a significant risk premium if Factor 2 is included. Although they do not mention it, this can be interpreted as evidence in favour of the earlier contention that price-impacts are largely driven by depth, breadth and immediacy. It supports the finding in Fleming [43] that price-impacts are correlated with all the measures they use.

Porter [72] provides a different methodology for testing the correlations between measures. Using the Pastor & Stambaugh return-reversal measure, the illiquidity ratio and the bid-ask spread, they compute a time series of innovations for each of these variables. They then subject the innovation series to a correlation analysis. They argue that since the innovations capture the additional information impacting a particular measure at a specific time, then if the variables measure the same “feature of aggregate liquidity then we would expect the innovations to be correlated”. This methodology helps overcome the problem of spurious correlations which is generally found in non-stationary series. Indeed most of the measures suffer from this problem ([29] and [3]).

Overall, Porter [72] finds contradictory results to the other papers. They find that all the innovations are significantly positively correlated, all show negative spikes at the same dates, all carry a significant risk premium of between 2-5% pa and all of them look surprisingly similar when plotted. Porter [72] neglects to mention that an innovation series captures new information impacting on a measure. Thus if new information impacts on all the measures in the same way (which is expected since this how markets process news) but different aspects of liquidity only become recognizable with a cumulative effect (which is also reasonable as liquidity changes via a cumulative trading process) then one would expect high correlations in the innovations and mixed correlations between the measures themselves.

In an attempt at finding a pervasively-priced liquidity factor, Chen [26] conducts a PCA on 7 scaled, aggregate liquidity proxies and finds similarly low correlations to Porter [72]. These correlations, however, fall in line with the previous findings and with those in Eckbo *et al* [41].

Chen [26] compares, among others, the bid-ask spread, stock turnover, the illiquidity ratio and the return-reversal measure for all NYSE and AMEX common stocks by averaging the measures across all the stocks in his sample for each month and then de-trending his series. The de-trending is effected by multiplying the series by  $\frac{w_1}{w_t}$ , where  $w_t$  is the 24-month preceding average of the measure and  $w_1$  is the measure value at a fixed date in the past. Chen [26] undertakes the de-trending to avoid spurious results, as Eckbo *et al* [41] find a positive trend in the market-wide measures that is reflective of the growing liquidity of all markets. It is only relevant when one is conducting a market-wide analysis of liquidity.

Overall the correlations in the scaled-aggregate measures are low, except for the illiquidity ratio and the return-reversal pair which have a correlation of 0.82. This is somewhat expected as the measures fall into 3 classes: spread-based measures, trade-activity measures and 2 price-impact measures. The PCA extracts only 3 components which are not as open to interpretation as the results of the factor analysis in Chollete *et al* [29].

In general, while there is some support for the idea that different measures capture different aspects of liquidity, the results are by no means overwhelmingly in favour of this hypothesis. The problems surrounding spurious correlations complicate the search for a complete set of liquidity risk management tools. This is compounded by the fact that the results of the PCA and factor analysis (other than those conducted by Chollete *et al*) are unclear and the correlations, when they do indicate differences in the measures, seem to do so very marginally. High correlations between the measures persist and may continue to do so until a better methodology to compare the measures is established.

One of the ways to address at least part of the problem is to test the time series for auto-correlation and accordingly difference them until one finds a non-stationary series. Correlations can then be based on this differenced series without the worry of spurious results. Although difficult to interpret economically, correlations on differenced series have the benefit of highlighting stable and robust patterns between time series. Moreover unlike the error term in an innovation series as used in Porter [72], they do not capture “new” information but retain the signal of the initial level variable set.

Although this methodology may lead to greater clarity, its results may not

necessarily support the idea that different measures objectively capture different information. The high correlations between measures and inconclusive PCA results may still be attributable to the fact that there is strong evidence of commonality in liquidity.

If liquidity, like market returns are driven by aggregate market-wide liquidity then, in so far as different measures capture this aggregate co-variation, they will be correlated and any factor extracted from a PCA or a factor analysis will load heavily on all the measures.

The literature regards commonality in liquidity as a well-established fact. For example Acharya *et al* [2] find that the quoted spread, depth and effective spreads all co-move with market-wide and industry-wide liquidity. This commonality remains even after controlling for volatility and volume. Using a regression of individual liquidity measures against their market-wide counterpart, they find ample evidence of commonality. Acharya *et al* [2] argue that commonality in liquidity should exist because of commonality in trading activity. They point out that trading activity generally shows a market-wide response to general price swings which are realized in trade volume. Since dealer quotes are, in turn, heavily influenced by volume, changes in prices induce changes in a wide range of liquidity measures as they become more influenced by the prevailing trading climate. This together with herding behaviour among the large institutional traders results in commonality.

In implementing their famous liquidity Capital Asset-Pricing Model (CAPM) Acharya *et al* [3] also find compelling evidence of commonality. Indeed the basis of their model is that the required rate of return on a security is increasingly related to the covariance of individual illiquidity with market-wide illiquidity, decreasing in the covariance of security return and market illiquidity and decreasing in the covariance between security illiquidity and the market return. Commonality then seems to impact on standard asset-pricing and risk in a number of significant ways. Lee [61] for example even finds evidence in favour of commonality at the global level.

The preponderance of commonality in liquidity casts doubts on the ability of standard correlation analysis and even PCA (but less so with this methodology which only extracts unique variance) to differentiate between measures. While it seems that the different measures undoubtedly capture different aspects of liquidity, a better methodology, one that specifically controls for commonality, is needed to confirm this. As it stands the literature uses far too many measures with no consensus regarding any of them. Such a methodology would provide firm evidence that liquidity can only be measured with a wide range of measures that capture its many forms.

In the absence of a clear methodological framework for the testing of the measures, a theoretical discussion within the context of the definition of liquidity risk and the requirements in Section 3.6 can assist in discriminating between the array of measures. This is presented in Section 4.6 where the measures are critically compared to the integrated Liquidity-VaR models.

## 4.5 Liquidity-VaR Models

Although it seems increasingly clear that liquidity can only be measured with a wide-range of measures, the first prize will always be a completely integrated risk measure that captures all aspects relating to liquidity risk. In searching for this, the literature has made a number of concerted efforts at developing integrated Liquidity-VaR models.

Liquidity-VaR models represent the first attempt at integrating the complete definition of liquidity risk into a workable VaR framework. As discussed earlier, although different measures are important in that they highlight some of the different aspects of liquidity, they do so in a fragmented and uncoordinated manner. No single measure provides a coherent description of market-wide and trade-specific liquidation costs as represented by all the different aspects. It is the goal of L-VaR models to correct this.

Over time the literature has presented a number of different L-VaR model innovations, each showcasing a different integration and understanding of liquidity risk with different underlying assumptions. While the array of models seems diverse, the significant models can be broadly separated into 3 classes:

1. **Spread-Adjustment Models** — these models try to correct standard VaR by incorporating liquidity-related costs as proxied by the bid-ask spread and its variability. Most of these models use a parametric VaR framework to adjust for the spread.
2. **Parametric Adjustment Models** — models in this class adjust the standard inputs to a parametric VaR, like  $\Sigma$  and  $\mu$ , for liquidity-related transaction costs.
3. **Optimal Liquidation Strategy Models** — these models are the most cohesive and robust, they model the liquidation schedule of a trader and base their VaR on the theoretical trade proceeds.

### 4.5.1 Spread-Adjustment Models

#### 1. The Bangia, Diebold *et al* Model, 1998 [14]

The Bangia model is loosely based on an earlier model by Jarrow & Subramanian. It is the first L-VaR model that accounts for spread-related risks and transaction costs. The model is easy to implement and is premised on the idea that liquidity risk can be separated from price risk.

Although, in their paper, Bangia *et al* [14] address the importance of both endogenous and exogenous liquidity risk, they choose to focus solely on exogenous risk as captured in the spread. They argue that since the spread can be easily accounted for with readily available data, doing so maintains their model's simplicity and ease-of-use while providing sufficient accuracy.

Specifically Bangia *et al* [14] point out that the spread is sufficiently important to all market players to warrant proper specification. While they concede that price impacts become increasingly important when trade occurs outside the quoted depth, they point out that the loss in accuracy brought about by ignoring this makes up for the onerous data gathering needed to model it properly.

#### The Model

Let  $r_t = \ln(\frac{P_t}{P_{t-1}}) = \ln(P_t) - \ln(P_{t-1})$  be the 1-day logarithmic return of an asset.

If now  $r_t \sim N(\mu, \sigma)$  is Gaussian then it can be shown that  $Var(99\%, P_t) = P_t e^{(\mu_t - 2.33\sigma_t)}$ .

If it is, without loss of generality, assumed that  $\mu_t \approx 0$  (which is possible as one can always adjust the return distribution so that this is true) then:

$$P - VaR = P_t(1 - e^{-2.33\sigma_t}) \quad (4.7)$$

is the 99% worst expected loss on the portfolio.  $\sigma_t$  here can be estimated by GARCH or an EWMA regime as noted earlier. In South Africa  $P_t$  is usually the close price on day t but may also be the mid-price, depending on the pricing convention.

Bangia *et al* echo the discussion on Extreme Value theory in Section 3.5 and argue that such a parametric VaR excludes tail events due to both

general market conditions and liquidity risk. They hold that incorporating a liquidity adjustment into the above VaR would help capture these tail events. They thus define the exogenous cost of liquidity (COL) such that  $COL = \frac{1}{2}P_t(\bar{S} + \alpha\bar{\sigma})$  where  $\bar{S}$  is the average relative spread (as defined earlier) and  $\alpha\bar{\sigma}$  is some multiple of the spread volatility so that one covers 99% of the spread distribution.<sup>2</sup>

Since the distribution of the spread  $S$  is generally far from normal (see Figure 4.2), Bangia *et al* point out that one cannot rely on distributional assumptions to estimate  $\alpha$ . They find that  $\alpha$  ranges from 2 to 4.5 and that the greater the departure from normality the higher its value. Generally, however, the use of Excel's critical value function applied to the empirical distribution works as a good estimate of  $\alpha$ .

In order to arrive at their L-VaR model, Bangia *et al* make the important assumption that extreme events in returns occur concurrently to extreme events in the spread. They thus assume perfect correlation between  $S$  and  $r$  and in "...calculating the liquidity-risk adjusted VaR ... incorporate both a 99th percentile movement in the underlying and in the spread". This is shown in the Figure 4.3.

With this assumption they arrive at the following estimate of L-VaR with a correction factor for fat tails:

$$L - VaR = P_t[1 - \exp(\mu_t - \alpha\theta\sigma_t)] + \frac{1}{2}P_t(\bar{S} + \tilde{\alpha}\tilde{\sigma}) \quad (4.8)$$

$P_t \rightarrow$  mid or close price

$\mu_t \rightarrow$  expected log return of  $P$

$\bar{S} \rightarrow$  average relative spread

$\alpha \rightarrow$  qth percentile of log return distribution

$\theta \rightarrow$  correction factor for fat tails

$\sigma_t \rightarrow$  standard deviation of the log return

$\tilde{\alpha} \rightarrow$  qth percentile of relative spread distribution

$\tilde{\sigma} \rightarrow$  standard deviation of relative spread distribution

The fat tail adjustment,  $\theta$ , is based on the Student t-distribution. Bangia *et al* point out that for large samples the t-distribution converges to the normal curve and for any t-distribution:

---

<sup>2</sup> $\alpha$  is basically the 1% critical value of the empirical spread distribution

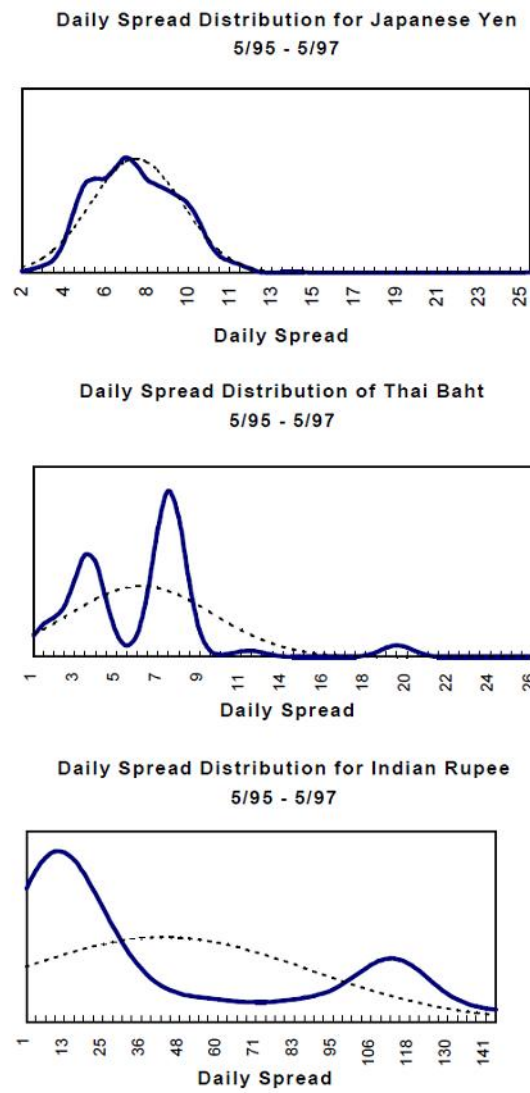


Figure 4.2: Spread Distributions [14].



$$\theta = 1 + \phi \ln\left(\frac{\kappa}{3}\right) \quad (4.9)$$

$\phi \rightarrow$  a constant that depends on the tail probability  
 $\kappa \rightarrow$  kurtosis of the empirical distribution

Given this, Bangia *et al* [14] estimate  $\phi$  by regressing estimates of the standard parametric VaR from equation 4.8 against historical simulation VaR estimates for the same empirical series. They get for the 99% VaR,  $\phi \approx 0.4$ .

They argue that such a regression against historical VaR estimates accounts for the non-normality and other pitfalls of parametric VaR without requiring a much larger data set. They point out that the model can be enhanced by using time-varying kurtosis and other dynamic estimates.

The model can also be extended as a portfolio VaR by either computing the correlation structure of the spreads, which may prove difficult, or by computing a portfolio weighted-average spread based on market-price weighted bids and asks and then applying their methodology to this weighted spread.

### Model Implementation

Bangia *et al* implement their model on FOREX markets using the Yen/\$ and the Thai Baht/\$ as test assets over May 97. During this period the Baht went from being pegged against the Dollar to a free-float currency. They calculate L-VaR both pre and post the crises and currency de-pegging.

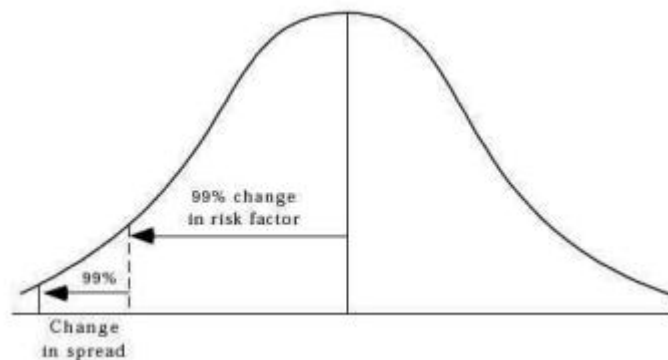


Figure 4.3: Parametric L-VaR [14].

Since their model encapsulates liquidity risk under COL, one can easily compute the additional risk due to liquidity by finding COL/L-VaR as a percentage. Bangia *et al* do this and find that prior to the East Asian crises liquidity risk accounts for 1.5% and 16% respectively for the Yen and the Baht. This changes to 1% and 5% post-crises. Bangia *et al* argue that this change reflects the fact that the floating rate and the heavy trading in the currency after the crises led to much of the spread risk being realized in the return. Prior to the de-pegging the spread accounted for more risk as this was the only way in which the market could express sentiment.

Unlike many of the other models which follow, Bangia *et al* backtest their model. They, however, only discuss the number of violations exhibited by their model and ignore the time trend of their VaR methodology. They also do not conduct any of the standard statistical tests associated with backtesting as discussed later in Section 5.2.2.

Overall they find that the number of VaR violations decrease once liquidity risk is brought into the VaR but that the difference in violations is smallest for portfolios consisting of US Bonds and G7 Currencies. This is expected as these portfolios tend to be highly liquid.

Le Saout [60] is one of the few papers to analyse the Bangia VaR. They implement the model on the Paris Stock Exchange. They get a value of 0.039 for  $\phi$  and estimate the proportion of L-VaR accounted for by liquidity. Interestingly they find that some illiquid stocks with small market capitalizations have as much as 34.96% of their L-VaR accounted for as liquidity risk.

In order to test this relationship between liquidity risk and market capitalization more thoroughly, Le Saout [60] regress COL for each stock against its market capitalization. Their regression model has an  $R^2$  of 13.9% and indicates a strong negative correlation.

## Critique

Most papers which reflect on the Bangia model, while finding the model easy to implement, generally question its underlying assumptions.

Le Saout [60] for example finds that extreme events in spreads and in returns are not well coordinated. The Bangia model thus underestimates risk. They also question the fact that the model ignores endogenous liquidity risk and propose a similar model based on the Average-Weighted Spread (AWS) in order to correct for this. Using the AWS, they find that the proportion of

VaR attributable to liquidity rises from 3.8% to 20.9%. This is compelling evidence for the importance of price impacts.

While this method seems more accurate, calculating the AWS (see Addendum A) is relatively difficult, as it relies on order book information based on the normal market size (NMS)<sup>3</sup>, which is not readily available at any decent frequencies in most exchanges.

Loebnitz [65], however, agrees with the concept underlying their argument and point out that the neglect of price impacts is unrealistic and that the assumption of a perfect correlation between spreads and returns could severely understate true risk, especially in times of stress.

They highlight a “structural inconsistency” in the Bangia model. As it stands, the model calculates the value after liquidation at the end of the forecast horizon and assumes a block sale at  $P_t$ . This forecast is then subjected to the spread adjustment by adding COL to the standard VaR metric.

Quite correctly, Loebnitz [65] contend that the liquidation value based on spread volatility should be modelled first and then a forecast applied to this spread-adjusted liquidation value. In this way the model is more consistent with the price formation process as it captures the effect of the spread on liquidation value not on the forecast value.

Loebnitz [65] thus propose an adjusted model which they, unfortunately, do not test.

$$L - VaR = P_t[1 - \exp(\mu_t - \alpha\theta\sigma_t)] + \frac{1}{2}P_te^{\mu_t - \alpha\theta\sigma_t}(\bar{S} + \tilde{\alpha}\tilde{\sigma}) \quad (4.10)$$

Overall, however, much of the literature refers to the Bangia *et al* model as a benchmark model. They point out that it incorporates a quick and easy adjustment to standard VaR that models at least part of liquidity risk. Indeed Loebnitz [65] argue that the model can be easily implemented in any market, like Bond markets, Swap markets, etc. where the spread accounts for a greater proportion of liquidity risk than do price impacts.

---

<sup>3</sup>The NMS of an asset is defined differently in different exchanges, but usually represents the minimum number of securities at which the market-maker must quote firm bids and asks. A trader can usually trade relatively easily at the NMS without moving the market.

## 2. Ohsawa and Muranaga, 1998 [69]

Around the same time that Bangia *et al* published their spread-adjustment model, Ohsawa & Muranaga published a very similar adjustment for Monte Carlo VaR.

Muranaga *et al* [69] hypothesise 3 different price simulations that account for different liquidity risk elements. The 1st process accounts for intra-day price movements, the 2nd for intra-day variability in the spread and the 3rd for market price impacts.

### The Model

#### 1. Intra-day Price Movements

Assuming that the volume-weighted average price (VWAP) of an asset follows a log-normal stochastic process, then one can model the expected execution price  $P_{Ex}$  as:

$$P_{Ex} = P_{VWAP}^0 e^{(\sigma_{VWAP} \epsilon_a \sqrt{t})} + \sigma_H \epsilon_b \quad (4.11)$$

$P_{Ex} \rightarrow$  expected execution price at end of holding period

$P_{VWAP}^0 \rightarrow$  VWAP on the risk evaluation day

$\sigma_{VWAP} \rightarrow$  observed historical VWAP volatility

$\sigma_H \rightarrow$  standard error of daily price standardized by VWAP distribution

$\epsilon_{a,b} \rightarrow$  standard, independent, normal variates

$t \rightarrow$  the holding period in days

#### 2. Spread Volatility

By modelling the Bid and Ask prices separately one can arrive at a model of the mid-price:

$$P_{Bid} = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) - \frac{1}{2} f(u) \quad (4.12)$$

$$P_{Ask} = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) + \frac{1}{2} f(u) \quad (4.13)$$

$P_M^0 \rightarrow$  mid-price on risk evaluation day  
 $\sigma_M \rightarrow$  standard deviation of mid-price  
 $\epsilon \rightarrow$  standard, independent, normal variate  
 $u \rightarrow$  standard, independent, uniform variate  
 $f \rightarrow$  empirical probability density function of the bid-ask spread,  
 based on a historical simulation of bid and ask prices over the  
 last year

## 2. Market Price Impacts

Muranaga *et al* [69] construct a price simulation which attempts at accounting for price impacts as well. Let  $\lambda$  denote the market price impact, being the sensitivity of the bid and ask prices to the volume traded. Muranaga *et al* [69] then estimate  $\lambda$  as the ratio of the price change (based on the difference between before-trade bid/ask quote and after trade bid/ask quote) to associated trade volume standardized by the normal market size, defined as the average daily trading volume. Based on this they arrive at the following simulation:

$$P_{Bid}^i = P_{Bid}^{i-1} \exp\left(\frac{-V^i g_\lambda(u)}{NMS}\right) \quad (4.14)$$

$$P_{Ask}^i = P_{Ask}^{i-1} \exp\left(\frac{V^i g_\lambda(u)}{NMS}\right) \quad (4.15)$$

where

$$P_{Bid}^0 = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) - \frac{1}{2} \mathbb{E}((f(u))) \quad (4.16)$$

$$P_{Ask}^0 = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) + \frac{1}{2} \mathbb{E}((f(u))) \quad (4.17)$$

$P_{Bid/Ask}^i \rightarrow$  after-trade bid/ask price  
 $g \rightarrow$  the historical probability density of  $\lambda$   
 $V^i \rightarrow$  traded volume of the  $i$ th transaction  
 $\epsilon \rightarrow$  standard, independent, normal variate  
 $u \rightarrow$  standard, independent, uniform variate

## Model Implementation

Muranaga *et al* [69] do not backtest their VaR estimates but merely compare their adjusted model to standard VaR and discuss the differences. They find that the differences to ordinary VaR become larger as the simulation becomes more complex. This, however, does not necessarily mean that the models become more accurate.

## Critique

Generally the Muranaga models are easy to implement as they require very little additional data. However, the models share the common pitfalls of standard Monte Carlo VaR.

Moreover the simulations themselves seem rather ad hoc and depend inordinately on distributional assumptions. Muranaga *et al* [69] provide no theoretical development for their price paths and do not explain why their models are better at capturing liquidity risk than standard VaR. Although using VWAP does, to some extent, capture price-related impacts and the spread-related risk, the model remains quantity-independent. It will thus always tend to under-estimate true risk.

### 3. François-Heude and Van Wynendale, 2002 [44]

François-Heude *et al* [44] extend the Bangia model in order to account for some its weaknesses. Specifically they take issue with the correlation assumption inherent to the model and the neglect of endogenous risk.

The Bangia *et al* model splits risk into market-related risk and exogenous liquidity risk, as proxied by the spread. This necessitates the separate modelling of the spread and price return and a correlation assumption relating the 2 models. François-Heude *et al* [44] try to overcome this by proposing a VaR based on a “theoretical bid obtained from the mid-price adjusted by half the average relative spread”. They also include a dynamic factor that captures how the mid-price moves relative to the effective spread and account for an endogenous liquidity risk by modelling the spread relative to the quantity being liquidated.

## The Model

In aggregate François-Heude *et al* [44] arrive at an L-VaR, as shown in 4.18, which applies the VaR methodology to a theoretical bid, adjusted by half the average relative spread,  $S_{pBL}^-$ . This formulation overcomes both the incorrect correlation assumption and the structural flaw of the Bangia model.

$$L - VaR_t = Mid_{BL_t} \left[ \left( 1 - \left( 1 - \frac{S_{pBL}^-}{2} \right) e^{-\alpha\sigma} \right) + \frac{1}{2} (S_{pBL,t} - S_{pBL}^-) \right] \quad (4.18)$$

$Mid_{BL_t}$  → the mid-price at time t  
 $S_{pBL,t} = \frac{1}{Mid_{BL_t}} (Ask_t - Bid_t)$  → the relative spread at time t  
 $S_{pBL}^-$  → the average relative spread  
 $\alpha$  → the qth percentile of the mid-return distribution  
 $\sigma$  → the standard deviation of the mid-return distribution

The above specification ignores price-impacts. In order to account for endogenous liquidity risk, François-Heude *et al* [44] favour the use of an adjusted relative spread  $S_{pt}(Q)$ , where Q is the quantity to be traded. They thus arrive at the following L-VaR:

$$L - VaR_t = Mid_{BL_t} \left[ \left( 1 - \left( 1 - \frac{SP(Q)}{2} \right) e^{-\alpha\sigma} \right) + \frac{1}{2} (S_{pt}(Q) - SP(Q)) \right] \quad (4.19)$$

$S_{pt}(Q) = Bid_t(Q) - Ask_t(Q)$  → the spread at time t adjusted to the quantity Q  
 $SP(Q)$  → the average over time of the spread adjusted to the quantity Q

A simple way to implement the above quantity-adjusted model is with the use of the Average Weighted Spreads (AWS), as discussed in the Le Saout [60] extension to the Bangia model. This allows one to calculate L-VaR adjusted to the Normal Market Size (NMS). However, François-Heude *et al* [44] want a model that is adjusted to any quantity Q, not only the NMS.

They thus propose using the 5 best order limits of the order book. They argue that these limits are usually sufficient to cover any quantity traded, thus the prices attached to them can be interpolated to find Q-adjusted prices. Moreover if the accumulated quantity available for trade at these limit prices is still insufficient to capture Q then one can use the Average Weighted Spread as the top bound of the interpolation. In this way the model captures market depth, breadth and to some extent endogenous liquidity risk.

François-Heude *et al* [44] have an algorithm for determining  $S_{pt}(Q)$ . Basically if Q is less than the quantity available at the bid or ask, then since the entire quantity can be traded within the spread,  $Bid_Q$  and  $Ask_Q$  is the Bid and Ask respectively. If, however, Q is larger than this quantity but remains less than the accumulated quantity available for trade at the 5 best order limits, then the Bids and Asks associated to Q are obtained by finding an AWS associated not to the NMS but to the quantity available at each of these 5 best limit order prices. They thus calculate an  $AWS_{Aski}$  and  $AWS_{Bidi}$  for each of the  $i=1, 2, \dots, 5$  limit orders and then interpolate based on these average-weighted spreads and their associated quantities to arrive at a spread-adjusted to Q. If, however, Q is greater than the accumulated quantity available at the 5 best orders, then they interpolate a bid and ask based on the NMS and the true AWS associated to it.

## Model Implementation

François-Heude *et al* [44] apply their model to a position that corresponds to NMS/2 and NMS/5 and use 5 days' worth of data to estimate parameters. In order to test their model, they estimate the Bangia model with 3 different parameter values for  $\alpha$ , and compare these to their model without endogenous risk.

They find that their model tends to be generally lower than the Bangia model, as it models the spread distribution less "aggressively". Interestingly they find the Bangia model to be extremely sensitive to changes in  $\alpha$ , indicating a source of model risk given the volatility of this parameter. They also compare their quantity-adjusted VaR, to VaR based on the NMS and find that the Q-adjusted models yield higher VaR estimates. Much of their implementation centres around proving that spread movements are not correlated to return movements and that the spread distribution is far from normal.

While François-Heude *et al* [44] do compare their model's time trends to that of the Bangia model, they do so without a proper backtesting framework to



highlight accuracy. They do not even quote the number of VaR violations. Moreover their comparison is only focused on a single, highly liquid stock for which quoted order book data is readily available on the Paris exchange.

## Critique

Overall the François-Heude *et al* [44] model, at least on the basis of theory, is a marked improvement on the Bangia model. Certainly the model without the quantity adjustment seems more realistic and is just as easy to implement as the original Bangia model. It avoids the overly-simplistic assumptions of the original model without necessitating drastic model changes.

The endogenous risk model, however, seems to be somewhat ineffective as it relies on data like the AWS and 5 best order book limits which is not readily available. Moreover since the order book changes so often, the model could be inordinately computationally intensive as it requires regular updating to remain relevant. Indeed Loebnitz [65] argues that even with the quantity-adjustment, liquidity risk is not really accounted for as price-impacts, as specified in Chapter 2, are ignored.

The model may also be prone to immense volatility in its forecasts as unlike other models, it uses the current level of the spread as an input.

## 4. Angelidis and Benos, 2006 [10]

The Angelidis *et al* [10] model aims at increasing the accuracy of the Bangia model by incorporating inventory-related spread costs and by indirectly modelling trade-size dependency. In this way they hope to include both endogenous and exogenous liquidity risk. Their work is more firmly grounded in market micro-structure theory than any of the preceding models.

### The Model

Let  $P_t$  denote the transaction price of a security at time  $t$  and  $X_t$  a trade direction indicator such that

$$X_t = \begin{cases} 1 & \text{if a buy} \\ -1 & \text{if a sell} \end{cases}$$

Then if  $\phi \geq 0$  represents the cost per share of the market-maker in supplying liquidity on demand,  $\theta$  models the information asymmetry between the market-maker and traders and  $\mu_t$  is the expected value of the stock at time  $t$  then one can model  $P_t$  as:

$$P_t = \mu_t + \phi X_t + \kappa(X_t \sqrt{V_t}) \quad (4.20)$$

where

$$\mu_t = \mu_{t-1} + \theta \sqrt{V_t}(X_t - \rho X_{t-1}) \quad (4.21)$$

$\kappa \rightarrow$  reveals whether order handling or inventory costs are more important

$V_t \rightarrow$  is the absolute number of shares traded at time  $t$

If now  $a_{X_t=1}$  and  $b_{X_t=-1}$  are the ask and bid prices associated to the trade  $X_t$  then using the Equations 4.20 and 4.21 with  $X_t = 1$  and  $X_t = -1$ :

$$a_{X_t=1} = \mu_{t-1} + \theta \sqrt{V_t}(1 - \mathbb{E}[X_t | X_{t-1}]) + (\phi + \kappa \sqrt{V_t}) \quad (4.22)$$

$$b_{X_t=-1} = \mu_{t-1} + \theta \sqrt{V_t}(-1 + \mathbb{E}[X_t | X_{t-1}]) - (\phi + \kappa \sqrt{V_t}) \quad (4.23)$$

with  $\rho = \mathbb{E}[X_t | X_{t-1}]$ . This specification yields the following model for the intra-day spread:

$$a_{X_t=1} - b_{X_t=-1} = 2[\sqrt{V_t}(\theta + \kappa) + \phi] \quad (4.24)$$

The above implied spread is positively correlated to the volume traded  $V_t$ , the adverse selection component  $\theta$  and the cost component  $\phi$ .

The relationship to  $\kappa$  depends on its sign, however. Angelidis *et al* [10] point out that if  $\kappa < 0$  then the order handling component dominates the inventory cost and the cost component decreases in  $V_t$ . They attribute this to possible economies of scale in trading. Both equations 4.20 and 4.21 can be estimated with the use of Generalized Methods of Moments (GMM), which is discussed briefly in Appendix B.

Now given the parametric spread and the Bangia formulation, Angelidis and Benos arrive at the following L-VaR model:

$$L - VaR = VaR + [(\theta + \kappa)\sqrt{V_t^{\alpha'}} + \phi] \quad (4.25)$$

where  $V_t^{\alpha'}$  is the  $\alpha'$  quantile of the traded volume distribution.

The model captures exogenous liquidity costs through the spread adjustment but also accounts for endogenous risk that is implied by  $V_t$ :

$$\text{Endogenous Liquidity Risk} = (\theta + \kappa)\sqrt{V_t} + \phi - (\phi + \kappa)\sqrt{\bar{V}_t} - \phi \text{ if } V_t \geq \bar{V}_t \quad (4.26)$$

### Model Implementation

The approach in Angelidis *et al* [10] is an improvement on the François-Heude and Van Wynendale model as it explicitly examines the relationship between the quantity to be liquidated and the costs of trading. Moreover the model can be implemented with readily available trade data instead of difficult to obtain order book data.

Angelidis *et al* [10] apply their model to the Athens Stock Exchange at half-hour tick intervals, using only Bid and Ask prices over June 2002 to 30 December 2002.

They provide an interesting analysis of the changes in the parameters intra-day and across longer time periods and implement a 99% and 95% VaR, comparing them to a standard parametric VaR. They separate their analysis between large cap Top 20 index stocks and smaller Top 40 stocks. They also back-test their methodology using the Kupiec conditional coverage test which is discussed later in Section 5.2.2. This, however, does not encompass the totality of backtesting analysis.

As expected they find that large cap stocks have a lower % of VaR explained by liquidity, while the small cap stocks have higher liquidity risks.

The back-testing results provide compelling evidence in favour of L-VaR as in all cases L-VaR has violations which are closer to the expected percentage. This is especially true for large cap stocks.

### Critique

Overall the Angelidis *et al* [10] model provides a robust, framework for the inclusion of liquidity risk into VaR. Not only does the model allow for more

accurate VaR forecasts but as it requires estimation of inventory-related costs and information asymmetries, it also provides an insightful analysis on the state of the overall market liquidity. It thus comes closest to the broad descriptive measures encountered earlier.

Beyond this, unlike earlier models, the Angelidis framework can be relatively easily implemented with readily available data. The model provides a holistic treatment of liquidity as it indirectly incorporates quantity effects through the spread-trade relationship.

While Loebnitz [65] largely agree with the above, they point out that the model crucially depends on a structural inventory framework which is not easily extended to other markets. If the price-change model fails to hold then the entire L-VaR model fails.

#### 4.5.2 Parametric-Adjustment Models

##### 1. Al Janabi's Multi-Liquidation Horizon Model, 2009 [5]

Al Janabi [5], like many of the other models in this section, consider a statistical adjustment to the mean and variance inputs to parametric VaR in order to account for liquidity risk. Specifically their adjustment accounts for the fact that illiquid securities are generally tightly held and traded less frequently than other securities in a portfolio.

In a method similar to the time-scaling of  $\sigma$ , Al Janabi propose a methodology that adjusts the variance input for each security's expected trade date. Unlike the scaling methodology, however, the Al Janabi adjustment does not assume that all securities are sold at the same time.

##### The Model

Assume that a position in a security is closed-out linearly over  $t$ -days. If each day's security returns are independent and identically distributed then the variance of the  $t$ -day loss,  $\sigma_{adj}^2$ , is merely the sum of the variance of each day's loss so that  $\sigma_{adj}^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_t^2$ , where  $\sigma_i^2$  is the variance of the loss on day  $i$ . Using the fact that returns are identically distributed then  $\sigma_{adj}^2 = t\sigma_1^2$ .

Now given linear liquidation and assuming that the variance of losses decreases linearly in  $t$ , Al Janabi [5] model  $\sigma_{adj}^2$  by scaling the variance of each

day's loss by  $\frac{t-i}{t}$  to get:

$$\sigma_{adj}^2 = [(\frac{t}{t})^2 \sigma_1^2 + (\frac{t-1}{t})^2 \sigma_1^2 + (\frac{t-2}{t})^2 \sigma_1^2 + \dots + (\frac{1}{t})^2 \sigma_1^2] \quad (4.27)$$

which can be re-expressed with  $\sigma^2 \equiv f(t)$  as:

$$\begin{aligned} \sigma_{adj}^2 &= f[(\frac{t}{t})^2 + (\frac{t-1}{t})^2 + (\frac{t-2}{t})^2 + \dots + (\frac{1}{t})^2] \\ &= f[(\frac{(2t+1)(t+1)}{6t})^{\frac{1}{2}}] \end{aligned} \quad (4.28)$$

so that the adjusted L-VaR must be as below for a security and as shown in 4.30 for a portfolio

$$L - VaR_{adj} = VaR \sqrt{\frac{(2t+1)(t+1)}{6t}} > VaR \quad (4.29)$$

$$L - VaR_{P_{adj}} = \sqrt{|L - VaR_{adj}|^T | \rho | |L - VaR_{adj}|} \quad (4.30)$$

Al Janabi [5] hold that L-VaR should reflect residual market risk due to liquidity. Moreover they propose that t be estimated as the total trading position size divided by the daily average trading volume in that security.

Although they implement their model on selected Gulf markets, they neither back-test it nor compare it to different L-VaR models.

### Critique

Overall the model is a very weak alternative to the standard time-scaling technique used to adjust VaR. It maintains the same unnecessary assumptions regarding return independence and variance additivity.

At best the model merely provides a method by which one can extend the scaling adjustment to individual securities in a portfolio, given that they each

may have different times to liquidation. It provides no theoretical argument supporting the idea that such an adjustment accounts for liquidity risk.

The model ignores price impacts, spread-related risk and even price risk to some extent. It is expected that the Al Janabi [5] model would substantially underestimate liquidity risk.

## 2. Shamroukh's Covariance Scaling Model [76]

Shamroukh propose a similar but more robust methodology to the Al Janabi model to account for liquidity risk in VaR. While they also use a scaling methodology, their scaling adjustment is based on a model of a trader's liquidation behaviour.

Shamroukh [76] consider a number of cases, progressing from one asset and one risk factor (being a standard Brownian motion that models risk) to non-uniform liquidation across multiple assets and risk factors. At each step the algebra becomes more complicated but the underlying methodology remains the same.

### The Model

#### 1. One asset and One Risk Factor:

Consider a portfolio of one asset  $S$  with an amount of  $X_0$  invested in it.

Suppose that  $S$  is only impacted by a single risk factor  $W$  that is generated by a standard Weiner process,  $Z$ , and define a liquidation schedule as a vector of trade dates and trade amounts.

Let  $T$  be the end of the holding period and  $\Delta t = T/n$  be the number of uniformly spaced trade dates at each  $i\Delta t$  for  $i = 1, \dots, n$  with  $S_i$ , the price of  $S$  at each trade date.

Assuming uniform liquidation across time, the remaining position at each time step  $i$  is  $X_0(1 - \frac{i}{n})$ .

Now if  $\Delta \ln(W_i) = \sigma \Delta Z_i$  with  $\Delta Z_i = \varepsilon \sqrt{\Delta t}$  for  $\varepsilon \sim N(0, 1)$ , denotes the change in the underlying risk factor that drives  $S$ , then one can derive the mean and variance of the portfolio value at time  $T$ ,  $V_{P,T}$ , as follows:

The cash-equivalent value of the portfolio at  $i = n$  is  $V_{P,T} = \sum_{i=1}^n \frac{X_0}{n} S_i$  as at each time step a quantity of  $\frac{X_0}{n}$  is sold at a price  $S_i$ .

Setting  $\Delta S_i = S_i - S_0$  then:

$$\begin{aligned} V_{P,T} &= \sum_{i=1}^n \frac{X_0}{n} (S_0 + \Delta S_i) \\ &= X_0 S_0 + \frac{X_0}{n} \sum_{i=1}^n \Delta S_i \\ &= X_0 S_0 + \frac{X_0}{n} \sum_{i=1}^n M \Delta W_i \end{aligned} \quad (4.31)$$

for  $M = \frac{\Delta S_i}{\Delta W_i}$ , if we assume that  $S_i$  changes only in response to changes in  $W$ .

Using the approximation  $\Delta W_i \approx W_0 \Delta \ln(W_i)$  then

$$\begin{aligned} V_{P,T} &= X_0 S_0 + \frac{X_0}{n} M \sum_{i=1}^n W_0 \Delta \ln(W_i) \\ &= X_0 S_0 + \frac{X_0 W_0 M}{n} \sum_{i=1}^n \Delta \ln(W_i) \end{aligned} \quad (4.32)$$

as  $\Delta \ln(W_i) = \Delta Z_i$  by definition.

Setting now  $\sigma_{P,T}^2$  as the variance of  $V_{P,T}$  then

$$\sigma_{P,T}^2 = \left( \frac{X_0 W_0 M}{n} \right)^2 \text{Variance} \left( \sum_{i=1}^n \Delta \ln(W_i) \right) \quad (4.33)$$

which is a function of the covariance matrix of  $\Delta \ln(W_i)$  for  $\epsilon_i \sim N(0, 1)$  and  $i = 1, \dots, n$ .

Thus one arrives at

$$\sigma_{P,T}^2 = \left( \frac{X_0 W_0 M}{n} \right)^2 \sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j}^2 \quad (4.34)$$

where  $\sigma_{i,j}^2$  is the covariance between  $\Delta \ln(W_i)$  and  $\Delta \ln(W_j)$ .

After lengthy derivations Shamroukh [76] arrive at

$$\sigma_{P,T}^2 = (X_0 W_0 M)^2 \sigma^2 \Delta t \left( \frac{(n+1)(2n+1)}{6n} \right) \quad (4.35)$$

This yields an adjusted L-VaR of

$$VaR = (\text{Standard VaR}) \left( \frac{(n+1)(2n+1)}{6n^2} \right)^{\frac{1}{2}} \quad (4.36)$$

which has the property that  $\lim_{n \rightarrow \infty} L - VaR = (\frac{1}{3})(\text{Standard VaR})$ .

Without going into the details of the derivation (which is much the same as the above) for each of the cases, we present the Shamroukh L-VaR in each case below. The case of non-uniform liquidation over risk factors is left out given that such a liquidation merely complicates the derivation but adds little to the analysis.

## 2. Multiple Assets and Risk Factors

$$L - VaR = \sqrt{Q' \Sigma Q} = \left( \frac{(n+1)(2n+1)}{6n^2} \right)^{\frac{1}{2}} (\text{Standard VaR}) \quad (4.37)$$

where  $Q$  is a vector of portfolio weights and  $\Sigma$  is the variance-covariance matrix of the asset returns.

## 3. Endogenous Liquidity Risk

Assuming that prices are affected by trade size via a function  $f$  such that  $S_L = f(S_i, \alpha, \beta, \text{Trade size})$  denotes the liquidation price of the asset.

Shamroukh [76] assumes the following specification:  $S_L = S_i e^{\alpha + \beta \text{Size}}$  where  $\alpha$  captures exogenous liquidity risk,  $\beta$  endogenous liquidity risk and  $\alpha, \beta < 0$ .

This yields an L-VaR of:

$$L - VaR = \sqrt{ZWZ'} + \text{Mean Adjustment} \quad (4.38)$$

$$\text{Mean Adjustment} = \sum_{j=1}^m X_j S_j (e^{(\alpha_j + \beta_j \frac{x_j}{n})} - 1) \quad (4.39)$$



where  $Z$  is the  $m \times 1$  vector of liquidity cost adjusted VaR maps such that  $z_j = \sum^{p_j} X_{i,0} M_{i,j} W_{j,0} e^{(\alpha_i + \beta_i \frac{X_i}{n})}$ .

Here  $X_i$  is the holding in the  $i$ th asset and  $p_j$  is the number of assets in the portfolio with exposure to risk factor  $W_j$  for  $j = 1, \dots, m$ . In this setting different assets are correlated to different risk factors.

### Critique

Shamroukh [76] do not implement their model and although it presents a more robust, statistical adjustment to parametric VaR which remains easy to implement, it suffers from the same shortcomings as the Al Janabi [5] model.

Firstly although the model tries to account for liquidity risk, its formulation of the supply curve and price formation process is overly simplistic and ignores spread-related risk. The risk factors all follow standard uncorrelated Brownian motions, which necessarily preclude the idea that in times of stress risk factors tend to become increasingly correlated and non-normal.

Secondly the model's assumptions regarding the liquidation process is not representative of the dynamic trading that actually takes place in the market. Indeed the liquidation schedule is completely independent of the trade price and price impacts are really ignored.

### 3. Berkowitz's Price-Elasticity Model, 2000 [16]

Berkowitz [16] derive a liquidity adjusted distribution from which L-VaR can be inferred. Unlike the preceding parametric-adjustment models, however, it is not based on an assumed liquidation schedule and fixed price process but on an optimal liquidation schedule that is derived from the price process.

#### The Model

Let  $p_t$  be the price of an asset at time  $t$  and  $q_t$  the associated trade size which is related to  $p_t$  by  $p_t = p_{t-1} + x_t - \theta q_t$ . Here  $x_t$  represents an exogenous market factor and  $\theta q_t$  represents the price impact related to the trade.

Now Berkowitz [16] assume that a trader facing such a price process will

choose  $(q_t)_{(t \in T)}$  that solves the following optimization problem:

$$\max \mathbb{E}_t \left[ \sum_t^T p_t q_t \right] \quad (4.40)$$

subject to  $\sum_t^T q_t = M_t$  where  $M_t$  is the total liquidation holding.

In this way the trader tries to maximise trade revenue by changing the amount traded up to  $t=T$ .

The optimal solution to this problem is derived by Bertsimas & Lo [18] under the assumption that  $x_t$  represents rational reactions to new information or preference shocks as:  $q_t^* = \frac{M_t}{T}$ . The optimal solution then calls for linear and constant liquidation from  $t$  to  $T$ .

Given this, using historical portfolio values and net flows,  $\theta$  can be estimated from the regression:

$$p_{t+1} - p_t = \alpha + x_{t+1} - \theta q_t^* + \varepsilon_t \quad (4.41)$$

Moreover the portfolio mean and additional variance due to liquidity can then be shown to be:

$$\mathbb{E}[y_{t+1}] = Q_t'(p_t + \mathbb{E}[x_{t+1}] - \theta \mathbb{E}[q_t^*]) \quad (4.42)$$

$$\text{Liquidity Variance} = Q_t'(\text{Variance}(\theta q_t))Q_t \quad (4.43)$$

where  $Q_t$  is the  $N \times 1$  vector of portfolio positions,  $y_t$  is the portfolio value at time  $t$  and  $x_t$  can be taken to be the return on the market index.

The portfolio liquidity VaR at time  $t$  can be derived from the above by forecasting at time  $t$  the  $t+1$  portfolio mean and variance and then using them in a parametric VaR framework.

Alternatively the entire one-step ahead distribution can be forecasted by using the fact that  $y_{t+1} = Q_t'(p_t + x_{t+1} - \theta q_t^*)$  and the inversion formula for characteristic functions to arrive at:

$$F(y_{t+1}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-isk}}{is} \mathbb{E}[e^{isQ_t'(p_t + x_{t+1})}] \cdot \mathbb{E}[e^{(is - \theta Q_t' q_s)}] ds \quad (4.44)$$

which can be solved by numerical methods and has the important benefit that higher-order moment information is preserved in the VaR estimate. If, however,  $f$  is taken from some tractable family of distributions fit to past data, then the equation can be solved analytically to arrive at a distribution for  $y$ .

### Model Implementation

The Berkowitz [16] model is better suited to a portfolio of securities for which actual net flows and values are recorded through time. Indeed the model was implemented for 4 different mutual funds: aggressive growth, growth, growth and income and precious metals.

The model was implemented within a normal parametric framework and a one-step ahead adjusted distribution function based on the above was derived from which liquidity VaR was inferred.

Berkowitz [16] find that the L-VaR of the aggressive funds is generally higher than that of the other mutual funds. No other analysis or discussion is conducted around the model.

### Critique

The Berkowitz model is, within bounds, easy to implement and uses, for asset managers at least, readily available flow data. The liquidity coefficient  $\theta$  captures price impacts and spread moves and thus the model captures the important aspects of liquidity risk.

Solving the distribution equation analytically can be problematic but this is largely overcome by assuming a normal distribution for portfolio returns.

Overall, however, the model lacks depth and detail. Like the other parametric-adjustment models, it does not offer a very rich liquidity model and seems rather simplistically based on a poor optimal liquidity strategy. If the underlying Bertsimas and Lo formulation and assumptions proves false then the entire VaR framework proves false. In particular the assumption in Bertsimas & Lo [18] that informed trading is independent of noise trade is questionable as generally market practitioners cannot easily distinguish between the two and take both at face value, adjusting their behaviour, particularly in times of stress, to the market trade.

### 4.5.3 Optimal Liquidation Strategy Models

Unlike the spread-adjustment and parametric-adjustment models, the models that follow derive their notion of L-VaR by specifying a supply curve process and a price-impact model and then using both to derive an optimal liquidation strategy as a reference point for the modelling of transaction costs.

Optimal liquidation strategy (OLS) models aim at capturing more of the significant aspects of liquidity risk. As a result they come closer to the original definition cited earlier and are formulated with the specific aim of combining both exogenous and endogenous liquidity-related transaction costs.

All of the OLS models are characterized by their precise modelling of price impacts and their mathematical tractability. They attempt at uniquely associating liquidity risk to the liquidation position in a way that includes general market risk as well.

#### 1. Jarrow & Subramanian's Liquidity Discount Model, 2001 [55]

The Jarrow & Subramanian liquidity discount model derives an optimal liquidation strategy in order to determine the value-under-liquidation of a given position. This value-under-liquidation is used to derive liquidity-adjusted portfolio profits and losses which are then used in the computation of a liquidity-adjusted VaR. In this way a standard VaR formulation is applied to a liquidity-adjusted return distribution.

In doing this Jarrow *et al* [55] assume that traders try to maximise the expected liquidation value of a position subject to an exogenously determined investment horizon  $T$ , a random permanent market-price impact and an imposed stochastic supply curve.

## The Model

Given a mid-price,  $p(t)$ , process of

$$dp(t) = p(t)(\alpha dt + \sigma dW(t)) \quad (4.45)$$

and a stochastic quantity discount of  $c(s)$  then the transaction price for the sale of  $s_i$  units at  $t_i$  can be modelled as  $p(t_i^+) = c(s_i)p(t_i)$  for  $c(s_i) \in [0, 1]$  where  $t_i^+$  denotes the time just before the trade is executed.

Let  $\Delta(s_i)$  denote the execution lag between the order placement at  $t_i$  and execution at  $t_i + \Delta(s_i)$  and  $(t_i, s_i)$  be a trading strategy, a double of transaction quantities and times such that  $t_n = T$  and  $\sum s_i = S$ , the total quantity to be liquidated. Any trader facing such an economy has the problem of choosing  $(t_i, s_i)$ , an admissible trading strategy, that fulfils:

$$\text{Max}[\mathbb{E}_0[\sum_{i=1}^n s_i c(s_i) p(t_i + \Delta(s_i)) e^{-r(t_i + \Delta(s_i))}]] \quad (4.46)$$

$r \rightarrow$  the risk free rate

$c(s_i) \rightarrow$  the price impact trade discount

$(\Delta(s_i)) \rightarrow$  the execution lag process

subject to the price process:

$$p(t_i + \Delta(s_i)) = p(0) \exp\left\{\left[\alpha - \frac{\sigma^2}{2}\right](t_i + \Delta(s_i)) + \sigma(W(t_i + \Delta(s_i)) - W(t_i))\right\} \quad (4.47)$$

Both  $c(s_i)$  and  $\Delta(s_i)$  can be chosen to be deterministic.

By letting  $u(p, s, t)$  denote the value strategy and by solving equation 4.46 as an impulse control problem, Jarrow *et al* [55] find the optimal trading strategy for a frictionless economy and for an economy that includes liquidity risk.  $u^*(p, s)$  denotes the optimal proceeds from such an optimal strategy.

In the case of no liquidity risk:

$$u^*(p, s) = \begin{cases} Sp & \text{if } \alpha \leq r \\ Sp e^{(\alpha - r)T} & \text{if } \alpha > r \end{cases}$$

Thus when there is no liquidity risk it is most optimal to sell everything immediately. This supports the discussion in Section 2.3.4.

However for an economy with liquidity risk and assuming what Jarrow *et al* [55] refer to as the “economies of scale condition” <sup>4</sup> holds, the following is true:

$$u^*(p, s) = \begin{cases} Spc(s)e^{[(\alpha-r)\Delta(s)]} & \text{if } \alpha \leq r \\ Spc(s)e^{[(\alpha-r)(T+\Delta(s))]} & \text{if } \alpha > r \end{cases}$$

In this case the liquidity proceeds are always less than the proceeds from a frictionless economy which seems highly reasonable.

Using the above, Jarrow *et al* [55] define  $p^*$  to be the “perfect liquidity price”. It is the price such that  $u^*(p^*, s) = u(p, s)$ , that is the expected proceeds from liquidation in an economy with liquidity risk equals the proceeds without liquidity frictions when trading at the current market price,  $p$ . This price can be found by noting that:

$$\begin{cases} Sp^* = Spc(s)e^{(\alpha-r)\Delta(s)} & \text{if } \alpha \leq r \\ Sp^*c(s)e^{(\alpha-r)T} = Spc(s)e^{(\alpha-r)(T+\Delta(s))} & \text{if } \alpha > r \\ \Leftrightarrow p^* = pc(s)e^{(\alpha-r)\Delta(s)} \end{cases}$$

$p^*$ , as defined above, is used to value a portfolio instead of  $p$  as it captures the “fair expected liquidation value”. Jarrow *et al* [55] base their return distribution from which they infer L-VaR on this theoretical price. They show that if the mid-price process follows the GBM shown in 4.45 then:

$$\begin{aligned} L - VaR = ps \mid & \left[ \alpha - \frac{\sigma^2}{2} \right] \mathbb{E}[\Delta(s)] + \mathbb{E}[\ln(c(s))] - 2[\sigma\sqrt{\mathbb{E}[\Delta(s)]} + \\ & \mid \alpha - \frac{\sigma^2}{2} \mid std[\Delta(s)] + std[\ln(c(s))]] \mid \end{aligned} \quad (4.48)$$

In this L-VaR the expected execution lag,  $\mathbb{E}[\Delta(s)]$ , replaces the fixed time horizon  $T$ . Thus the horizon is a fraction of the trade size as larger orders induce longer time horizons. Moreover increases in the expected trade discount  $\mathbb{E}[\ln(c(s))]$  or return volatility raises the VaR.

---

<sup>4</sup>This states that splitting an order into two trades and committing to consecutive trades immediately is always more costly than a block sale

## Critique

Although Jarrow *et al* [55] do not implement their model, at least theoretically, it has strong advantages over preceding models.

The model's methodological framework, specifically, is more robust and has a sound mathematical footing as a stochastic impulse control problem. Unlike the Berkowitz model, which also offers an optimal liquidation schedule, it specifically accounts for price impacts and even accounts for their randomness. The model offers rich detail coupled with a certain degree of conceptual simplicity.

Despite this, it does have its drawbacks with regard to practicality. Objective estimation of its input parameters, for instance, like the mean and standard deviation of the execution lag  $\Delta(s)$  and the corresponding quantity discount  $c(s)$  (although subjective estimates can be used) is difficult, particularly because these parameters are not observable.

The model also insufficiently accounts for the full spectrum of liquidity risk in that it only models permanent price impacts and ignores temporary, volume-related price effects. It also fails to properly account for spread-related costs and does so only indirectly via the price-impact function, whose form is not clearly specified.

In addition the economies of scale condition is unrealistic as traders routinely split their orders to reduce trading costs. The assumption contradicts the result in Çetin *et al* [23] that continuous trading of small amounts can eliminate liquidity costs. The economies of scale condition ignores the opportunity costs related to trading and is almost certainly not true of very large sales.

Nevertheless Loebnitz [65] finds the model laudable given its, at least tentative, price-impact formulation and general rigour. The model is a good showcase for the elegance of OLS VaR models and there have been several attempts at extending its application.

Botha [20] for instance aim at extending the standard model to account for a portfolio of assets. Using an alternative formulation of the model as given by equation 4.49 and assuming that  $\mu \simeq 0$ , they offer a heuristic derivation of a portfolio L-VaR.

$$L-VaR_t = ps\{\mu(\mathbb{E}[\Delta(s)] + \mathbb{E}[\ln(c(s))]) - CI(\sigma_E \sqrt{\mathbb{E}[\Delta(s)]} + |\mu| \sigma_{\Delta(s)} + \sigma_{\ln(c(s))})\}$$

(4.49)

$p \rightarrow$  equity fair-value mid-price  
 $s \rightarrow$  units of equity held  
 $\mu \rightarrow$  average asset return  
 $CI \rightarrow$  chosen confidence interval  
 $\sigma_E \rightarrow$  volatility of equity return  
 $\mathbb{E}[\Delta(s)] \rightarrow$  mean of the liquidity interval  
 $\sigma_{\Delta(s)} \rightarrow$  standard deviation of the liquidity interval  
 $\mathbb{E}[\ln(c(s))] \rightarrow$  mean of the liquidity discount  
 $\sigma_{\ln(c(s))} \rightarrow$  standard deviation of the liquidity discount

If now  $\mu \approx 0$  then equation 4.49 simplifies to

$$L - VaR_t = psCI[\sigma_E \sqrt{\mathbb{E}[\Delta(s)]} + \sigma_{\ln(c(s))}] \quad (4.50)$$

Now the standard VaR for a portfolio P can be expressed as  $VaR_{simple} = N(\mu T - CI\sigma_P\sqrt{T})$  where  $\sigma_P$  is the variance-covariance matrix of the assets held in the portfolio and CI is the required confidence interval. This simplifies to  $VaR_{simple} = N(CI\sigma_P\sqrt{T})$  under the assumption of small  $\mu$ .

Now for standard VaR:

$$\begin{aligned}
 L - VaR_{simple} &= N \cdot CI \sqrt{(w_A \ w_B) \begin{pmatrix} \sigma_A^2 & \sigma_A \sigma_B \rho_{AB} \\ \sigma_A \sigma_B \rho_{AB} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix}} \sqrt{T} \\
 &= N \cdot CI \sqrt{(w_A \sigma_A \ w_B \sigma_B) \begin{pmatrix} 1 & \rho_{AB} \\ \rho_{AB} & 1 \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix}} \sqrt{T} \\
 &= \sqrt{(Nw_A CI \sigma_A \sqrt{T} \ Nw_B CI \sigma_B \sqrt{T}) \begin{pmatrix} 1 & \rho_{AB} \\ \rho_{AB} & 1 \end{pmatrix} \begin{pmatrix} Nw_A CI \sigma_A \sqrt{T} \\ Nw_B CI \sigma_B \sqrt{T} \end{pmatrix}} \\
 &= \sqrt{(VaR_{simple}^A \ VaR_{simple}^B) \begin{pmatrix} 1 & \rho_{AB} \\ \rho_{AB} & 1 \end{pmatrix} \begin{pmatrix} VaR_{simple}^A \\ VaR_{simple}^B \end{pmatrix}} \quad (4.51)
 \end{aligned}$$

This leads Botha [20] to hypothesise the following for the Jarrow & Subramanian portfolio L-VaR:

$$L - VaR = \sqrt{(L - VaR_t^A \ L - VaR_t^B) \begin{pmatrix} 1 & \rho_{AB} \\ \rho_{AB} & 1 \end{pmatrix} \begin{pmatrix} L - VaR_t^A \\ L - VaR_t^B \end{pmatrix}}$$



(4.52)

The attractive factor of this formulation is that Botha implements the portfolio VaR model using realized portfolio trades, profits and losses. He compares a time series of the Jarrow & Subramanian L-VaR against standard VaR and even back-tests the model.

Botha [20] estimate the expected time to execution,  $\mathbb{E}[\Delta(s)]$ , as the simple average of the time between placing a bid/offer and then having it executed and the liquidity discount as the difference between the market value of a trader's position at the time of the bid/offer and the value under liquidation. They find that the L-VaR is far more volatile than the standard VaR across portfolios. In addition in many cases the back-testing results show the L-VaR to be far more accurate than the standard VaR. Overall they conclude that while implementing the Jarrow-Subramanian model is not simple, the portfolio data can be found and the potential for a far more accurate VaR forecast is great.

## 2. Almgren & Chriss Value-Under-Liquidation Model, 1999 [6]

Almgren & Chriss [6] provide one of the first complete and robust extensions of standard VaR to include all the major aspects of liquidity risk.

Using an optimal liquidation strategy framework, Almgren *et al* [6] derive the exogenous and endogenous costs involved in liquidation. The framework accounts for both temporary and permanent endogenous price effects by using two different price impact functions. Indeed it is the first model to derive alternate liquidation strategies for a variety of different price impact specifications.

### The Model

Consider a trader with  $X$  units of an asset who wishes to trade his entire position by some exogenously determined date  $T$  in a discrete-time setting.

Let  $\tau = \frac{T}{N}$  and  $t_k = k \cdot \tau$  for  $k = 0, 1, \dots, N$  be a discrete set of trading times. Then Almgren *et al* [6] define a trading strategy as the collection  $\{x_0, x_1, \dots, x_N\}$  of units held after trading at each  $t_i$  such that  $x_0 = X$  and  $x_N = 0$ .

If now  $\eta_k = x_{k-1} - x_k$  denotes the units traded at each  $t_k$  time step then  $x_k = X - \sum_{j=1}^k \eta_j = \sum_{j=k+1}^N \eta_j$ .

In addition suppose that the price of the security at the  $k$ th time step is  $S_k$  such that

$$\begin{aligned} S_k &= S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{\eta_k}{\tau}\right) \\ &= S_0 + \sigma\sqrt{\tau} \sum_{j=1}^k \xi_j - \tau \sum_{j=1}^k g\left(\frac{\eta_j}{\tau}\right) \end{aligned} \quad (4.53)$$

$\sigma \rightarrow$  volatility of asset returns  
 $\tau \rightarrow$  length of time interval between trade dates  
 $\xi_k \rightarrow$  a random draw from a  $N(0, 1)$  distribution  
 $g(\nu) \rightarrow$  permanent price impact function  
 $\frac{\eta_k}{\tau} \rightarrow$  average rate of trading over  $t_k - t_{k-1}$

Given the difference between the permanent and temporary price impacts, the actual transaction price of the  $k$ th trade is better modelled as  $\tilde{S} = S_{k-1} - h(\frac{\eta_k}{\tau})$  where  $h(\nu)$  is the temporary price impact function.

The idea behind  $\tilde{S}$  is that “short-term supply and demand imbalances” due to trade cause temporary price concessions which disappear once the trade pressure has eased. This is exactly as mentioned in Section 2.3.4.

Now given these quantity and price trajectories one can derive the total net value received (less liquidity costs) of trading from  $t = 0$  to  $t = T$  as:

$$\sum_{k=1}^N \eta_k \tilde{S}_k = X_0 S_0 + \sum_{k=1}^N (\sigma\sqrt{\tau}\xi_k - \tau g(\frac{\eta_k}{\tau})) x_k - \sum_{k=1}^N \eta_k h(\frac{\eta_k}{\tau}) \quad (4.54)$$

$X_0 S_0 \rightarrow$  the market value of the initial position  
 $\sigma\sqrt{\tau}\xi_k \rightarrow$  aggregate effect of return volatility  
 $\tau g(\frac{\eta_k}{\tau}) \rightarrow$  accumulated permanent price concessions  
 $\eta_k h(\frac{\eta_k}{\tau}) \rightarrow$  temporary price decline concessions

The total cost of the trade is then given by  $X_0 S_0 - \sum_{\eta_k} \tilde{S}_k$ , the difference between the initial market value and the proceeds from liquidation. This shortfall is a random variable and thus has a mean and variance:

$$\mathbb{E}[x] = \sum_{k=1}^N \tau x_k g\left(\frac{\eta_k}{\tau}\right) + \sum_{k=1}^N \eta_k h\left(\frac{\eta_k}{\tau}\right) \quad (4.55)$$

$$\text{Var}[x] = \sigma^2 \sum_{k=1}^N \tau x_k^2 \quad (4.56)$$

If additionally each  $\xi_k \sim N(0, 1)$  is independently identically distributed then the shortfall is also normally distributed with the above mean and variance.

Using the above specifications, Almgren *et al* [6] extend their model by assuming that traders wish to minimize  $U(x)$  where  $U(x) = (\mathbb{E}[x] + \lambda \text{Var}[x])$ .  $\lambda$  denotes traders' degree of risk aversion.

The optimization problem amounts to minimizing  $U(x)$  by choosing alternative trading trajectories subject to the price impact formulations and the boundary conditions  $x_0 = X$  and  $x_N = 0$ . Almgren & Chriss only optimize  $U(x)$  for the case of linear price impacts, while they do present a few non-linear formulations, only the non-linear case provides a neat closed-form solution to the optimization problem.

Amongst other models, Almgren & Chriss consider the specifications as shown in Table 4.1.

The estimation of the above parameters is not trivial and depends on intra-day, high frequency data which can be extremely volatile and changeable. The dynamic nature of the inputs also means that they should be re-estimated often to maintain the relevance of the model.

Assuming the linear case, Almgren & Chriss arrive, by direct substitution of  $g$  and  $h$  into Equation 4.54, at:

$$\mathbb{E}[x] = \frac{1}{2} \gamma x^2 + \epsilon \sum_{k=1}^N |\eta_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^N \eta_k^2 \quad (4.57)$$

where  $\tilde{\eta} = \eta - \frac{1}{2} \gamma \tau$ .

	Non-Linear	Linear
Permanent Price Impact	$g(\nu) = \gamma  \nu ^\alpha$	$g(\nu) = \gamma\nu$
Temporary Price Impact	$h(\nu) = \eta  \nu ^\beta$	$h(\eta_k) = \epsilon \text{sign}(\eta_k) + \frac{\eta}{\tau} \eta_k$

$\gamma \rightarrow$  a constant that represents the amount by which prices fall or rise, measured in (currency/unit)/unit. Thus if one sells  $n$  units, the price per unit is depreciated (appreciated) by  $n\gamma$  regardless of the time of trade.

$\xi \rightarrow$  fixed cost of trading, usually half the bid-ask spread, plus commissions and fees

$\eta \rightarrow$  transient, price-impact cost

$\alpha, \beta \in [0, 1] \rightarrow$  power-law exponents

Table 4.1: Price-Impact Functions [6]

Since  $\mathbb{E}[x]$  is convex  $\Leftrightarrow \tilde{\eta} > 0$  and since  $U(x)$  is convex  $\Leftrightarrow \lambda > 0$  then  $U(x)$  has a minimum for some  $x$ . This minimum can be determined by taking partial derivatives in  $x$  to get:

$$\begin{aligned} \frac{\delta U}{\delta x_j} &= 2\tau(\lambda\sigma^2 x_j - \tilde{\eta} \frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2}) = 0 \\ \Leftrightarrow \frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} &= \frac{\lambda\sigma^2}{\tilde{\kappa}} x_j = \tilde{\kappa}^2 x_j \end{aligned} \quad (4.58)$$

for  $\tilde{\kappa} = \sigma \sqrt{\frac{\lambda}{(\eta - \frac{\gamma\tau}{2})}}$ .

The above solves as a linear difference equation with  $x_0 = X$  and  $x_N = 0$ :

$$x_j = \frac{\sin h(\kappa(T - t_j))}{\sin h} X \text{ for } j = 0, \dots, N \quad (4.59)$$

$$\eta_j = \frac{2 \sin h(\frac{1}{2}\kappa\tau)}{\sin h(\kappa\tau)} \cos h(\kappa(T - t_{j-\frac{1}{2}})) X \quad (4.60)$$

where  $\sin h$  and  $\cos h$  are the hyperbolic sine and cosine functions with  $t_j = (j - \frac{1}{2})\tau$  for  $j = 0, \dots, N$ .

The L-VaR can now be derived, in a manner similar to the earlier parametric L-VaR adjustments, by using the standard parametric VaR formulation together with the adjusted mean and variance calculated using the optimal trajectory defined in Equations 4.59 and 4.60.

$$L - VaR_p = \alpha \sqrt{Var[x]} + \mathbb{E}[x] \quad (4.61)$$

The above defines the maximal expected loss for the trading strategy for a probability  $p$ . The parameter  $\alpha$  is determined using the inverse cumulative probability distribution function of the return process such that  $\mathbb{P}(R < \alpha) = p$ .

## Model Implementation

Almgren & Chriss [6] do not implement their model themselves but Loebnitz [65] undertakes an examination by comparing model forecasts for different price-impact function formulations. They also conduct a sensitivity analysis.

Applying the model to assumed parameter inputs and examining the effect of changes on both expected shortfall and liquidity VaR, Loebnitz [65] find that the expected shortfall (ES) values are higher than VaR forecasts across all the price-impact specifications. This is somewhat expected given the fact that, as discussed earlier, ES more accurately captures low-probability, tail losses.

In terms of sensitivity, Loebnitz [65] find that at larger holdings, the loss as forecasted by standard VaR is far lower than the loss forecasted by the Almgren & Chriss L-VaR. They argue that this is to be expected as when holdings are large the effect of a block sale, as assumed by standard VaR, is far more onerous on the liquidation value.

Interestingly, large increases in the exogenous time horizon  $T$ , reduces the differences between the different models, but at small  $T$ , the differences become more apparent. This merely represents the effect of an immediate block sale versus staggered sales – small  $T$  implies a block sale which induces larger liquidity costs.

While a rigorous back-testing procedure applied to the model's forecasts would shed more light on its accuracy, the above results, at least superficially, indicate the model's strengths in capturing liquidity risks.

## Critique

The Almgren & Chriss formulation requires onerous data warehousing and parameter estimations, yet its richness and depth lends itself to a far more accurate model of VaR than any of the previous models.

Unlike the Jarrow & Subramanian model, Almgren *et al* [6] include both temporary and permanent price impact functions. These are incorporated into the model derivation in a way that is general enough to allow users to

specify the most suitable price impact formulations for the asset at hand. In this way the model can be implemented for bonds, FOREX, equity markets, etc. by making minimal changes to its form. This is, however, subject to the impact functions being amenable to a tractable solution to the optimization problem. As it stands the model provides very little guidance in terms of estimating the onerous price impact parameters.

Overall the model neatly balances price risks and liquidity-related price effects by properly considering the actual trade-off faced by market participants in trading. The model can only really be judged, however, by its accuracy and practicability.

### 3. Hisata & Yamai Liquidity Risk Model, 2000 [49]

One of the drawbacks of the Almgren *et al* [6] model is its assumption of an exogenously determined trading horizon  $T$ . Generally traders enter the market and then re-select their trading horizon in order to minimize liquidation costs and price uncertainty.

Hisata & Yamai extend the Almgren & Chriss formulation to specifically account for this by deriving an optimal horizon given sales at a constant rate. Thus instead of setting  $\frac{\delta U}{\delta x} = 0$  to find the optimal liquidation schedule, they set  $\frac{\delta U}{\delta N} = 0$ . In this setting a given sales schedule implied by the constant rate of trade is assumed.

#### The Model

As with the previous model, suppose a trader begins with  $X$  units of an asset and wishes to liquidate his position over  $t_0 = 0$  to  $t_N = T$  by selling at a constant rate over  $N$  time periods.

A position is thus sold at each  $t_k = k\tau$  for  $k = 0, \dots, N$  with  $t_N = T = N\tau$  and  $\tau = \frac{T}{N}$ .

Here  $N$  is unknown but at each  $k$  an amount of  $n_1, \dots, n_N$  units are sold with  $\nu_k = \frac{n_k}{\tau}$ , representing the rate of trade in the  $k$ th sub-period.

Similar to Almgren & Chriss, Hisata & Yamai assume a linear price-impact function so that the market price  $S_k$  is given by the arithmetic walk:

$$S_k = S_0 + \sigma \sum_{j=1}^k \tau^{\frac{1}{2}} \xi_j + \mu t_k - \gamma(X - x_k) \quad (4.62)$$

Given this the transaction price of the  $k$ th trade is  $\tilde{S} = S_k - \varepsilon - \eta\nu_k$ , which includes the permanent price impact. Thus the total transaction cost of the trading strategy is

$$\begin{aligned}
C &= XS_0 - X\tilde{S} \\
&= -\sigma \sum_{k=1}^N \tau^{\frac{1}{2}} \xi_k x_k - \mu t_k \sum_{k=1}^N \tau x_k + \frac{1}{2} \gamma X^2 \\
&\quad + \varepsilon X + \left(\eta + \frac{1}{2} \gamma \tau\right) \sum_{k=1}^N \tau \nu_k^2
\end{aligned} \tag{4.63}$$

Given that  $\xi_k \sim N(0, 1)$  is independently, identically distributed then the mean and variance of  $C$  must be

$$\mathbb{E}[C] = -\mu \sum_{k=1}^N \tau x_k + \frac{1}{2} \gamma X^2 + \varepsilon X + \left(\eta + \frac{1}{2} \gamma \tau\right) \sum_{k=1}^N \tau \nu_k^2 \tag{4.64}$$

$$Var[C] = \sigma^2 \sum_{k=1}^N \tau x_k^2 \tag{4.65}$$

Supposing that the trader wishes to choose  $N$  so that  $U = \mathbb{E}[C] + r Z_\alpha \sqrt{Var[C]}$  is minimized with  $Z_\alpha$  being the upper  $100\alpha$  percentile of the normal distribution and  $r$  his cost of capital. Assuming sales at a constant speed then:

$$\mathbb{E}[C] = -\frac{1}{2} \mu \tau X(N-1) + \frac{1}{2} \gamma X^2 + \varepsilon X + \frac{\eta X^2}{\tau N} + \frac{\eta X^2}{2N} \tag{4.66}$$

$$Var[C] = \frac{1}{3} \sigma^2 \tau X^2 N \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{2N}\right) \tag{4.67}$$

$$\begin{aligned}
U &= -\frac{\mu \tau X(N-1)}{2} + \frac{\gamma X^2}{2} + \varepsilon X + \frac{\eta X^2}{\tau N} + \frac{\eta X^2}{2N} \\
&\quad + r Z_\alpha \sqrt{\frac{1}{3} \sigma^2 \tau X^2 N \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{2N}\right)}
\end{aligned} \tag{4.68}$$

U is minimized at  $N^*$  such that  $\frac{\delta U(N^*)}{\delta N} = 0$

$$\Leftrightarrow \frac{\delta U}{\delta N} = \frac{-\mu\tau X}{2} - \frac{\eta X^2}{\tau N^2} - \frac{\gamma X^2}{2N^2} + \frac{rZ_\alpha[(\frac{\sigma^2\tau X^2}{3})^{\frac{1}{2}}(1 - \frac{1}{2N^2})]}{2[N - \frac{3}{2} + \frac{1}{2N}]^{\frac{1}{2}}} = 0 \quad (4.69)$$

which can be re-arranged to obtain a 6-degree polynomial in N. This has no closed-form solution.

However, if  $\mu \approx 0$  which is reasonable in the short time space between trades and assumed in many of the models then:

$$\frac{(N^2 - \frac{1}{2})}{2[N - \frac{3}{2} + \frac{1}{2N}]^{\frac{1}{2}}} = (\frac{\eta}{\tau} + \frac{\gamma}{2}) \frac{\sqrt{3}X}{rZ_\alpha\sigma\sqrt{T}} \quad (4.70)$$

whose solution yields an optimal N and optimal  $T = N\tau$ .

L-VaR then in the discrete-time setting for optimal  $N^*$  is given by

$$L - VaR = Z_\alpha\sqrt{Var[C]} \quad (4.71)$$

Similar to Almgren & Chriss, Hisata *et al* [49] also develop a continuous-time model, which unlike the discrete-time model, has a closed-form solution for N.

In the continuous-time model, for a standard Brownian motion  $Z(t)$  and stock price path  $S(t)$ :

$$\tilde{S}(t) = S(0) + \mu t + \sigma Z(t) - \varepsilon - \eta\nu(t) - \gamma \int_0^t \nu(s)ds \quad (4.72)$$

When the speed of trade is constant,  $\nu(t) = \nu \Rightarrow dx = -\nu dt$  so that

$$\begin{aligned} -\int_0^T \tilde{S}(t)dx &= \nu \int_0^T \tilde{S}(t)dt \\ &= \gamma \int_0^T [S(0) + \mu t + \sigma Z(t) - \varepsilon - \eta\nu - \gamma \int_0^t \nu ds]dt \\ &= XS(0) + \frac{1}{2}\mu\nu T^2 + \nu\sigma \int_0^T Z(t)dt - \varepsilon\nu T \\ &\quad - \eta\nu^2 T - \frac{1}{2}\gamma\nu^2 T^2 \end{aligned} \quad (4.73)$$



where the term  $XS(0)$  follows from the fact that  $\gamma \int_0^T S(0)dt = S(0)\gamma T = S_0X$  as  $X = \int_0^T \gamma dt$ .

Moreover the cost of execution  $C$  and its mean and variance are

$$\begin{aligned} C &= XS(0) - \left(-\int_0^T \tilde{S}(t)dx\right) \\ &= -\frac{1}{2}\mu\nu T^2 - \nu\sigma \int_0^T Z(t)dt + \varepsilon\nu T + \eta\nu^2 T + \frac{1}{2}\gamma\nu^2 T^2 \end{aligned} \quad (4.74)$$

$$\mathbb{E}[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^2}{T} + \frac{1}{2}\gamma X^2 \quad (4.75)$$

$$Var[C] = \frac{1}{3}T\sigma^2 X^2 \quad (4.76)$$

Assuming that the trader wishes to minimize  $U = \mathbb{E}[C] + rZ_\alpha\sqrt{Var[C]}$  then, one arrives at the closed-form solution for  $T$ :

$$\frac{\delta U}{\delta T} = 0 \Leftrightarrow T = \left(\frac{2\sqrt{3}\eta X}{rZ_\alpha\sigma}\right)^{\frac{2}{3}} \quad (4.77)$$

L-VaR in the continuous-time model for linear-deterministic price-impacts is then

$$L - VaR = \left(\frac{2\eta\sigma^2 Z_\alpha^2 X^4}{3r}\right)^{\frac{1}{3}} \quad (4.78)$$

The strongest aspect of the Hisata & Yamai model is that it provides the scope for a variety of price-impact functions. The fact that the model optimizes  $N$  which is independent of the price impact instead of the sales schedule, makes it more amenable to different impact specifications.

In this regard, Hisata *et al* [49] extend their model by assuming a stochastic market impact and even a non-linear market impact.

If  $\eta_t = \eta_0 + \sigma_n Z_n(t)$  represents the temporary market impact coefficient at time  $t$  for a given market-impact volatility  $\sigma_0$  and a Stochastic Brownian

<b>Temporary price impact: linear, constant</b> <b>Permanent price impact: linear, constant</b>	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^2}{T} + \frac{1}{2}\gamma X^2$ $V[C] = \frac{1}{3}T\sigma^2 X^2$
<b>Temporary price impact: nonlinear, constant<sup>93</sup></b> <b>Permanent price impact: linear, constant</b>	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^{3/2}}{\sqrt{T}} + \frac{1}{2}\gamma X^2$ $V[C] = \frac{1}{3}T\sigma^2 X^2$
<b>Temporary price impact: linear, random<sup>94</sup></b> <b>Permanent price impact: linear, constant</b>	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$ $V[C] = \frac{1}{3}X^2 \left( T\sigma^2 + \frac{\sigma_\eta^2 X^2}{T} \right)$
<b>Temporary price impact: linear, random<sup>95</sup></b> <b>Permanent price impact: linear, constant</b>	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$ $V[C] = X^2 \left( \frac{1}{3}T\sigma^2 + \frac{\sigma_\eta^2 X^2}{T^2} \right)$
<b>Temporary price impact: linear, random</b> <b>Permanent price impact: linear, random</b>	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$ $V[C] = \frac{1}{15}X^2 \left( 5T\sigma^2 + 2\sigma_\eta^2 X^2 T + \frac{5\sigma_\gamma^2 X^2}{T} \right)$
<b>Temporary price impact: linear, random</b> <b>Permanent price impact: linear, constant</b> Correlation between temporary price impact and asset prices.	$E[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$ $V[C] = \frac{1}{3}X^2 \left( T\sigma^2 + \frac{\sigma_\eta^2 X^2}{T} - 2\sigma\sigma_\eta\rho X \right)$

Figure 4.4: Alternate Price Impact Formulations [65]

motion  $Z_n(t)$  then by assuming that the transaction price is independent of this coefficient and following the above derivation, Hisata & Yamai [49] derive  $\mathbb{E}[C]$  and  $\text{Var}[C]$  as disclosed in Figure 4.4.

The non-linear and stochastic price impact forms are especially important given the empirical findings that impacts are generally concave in trade size. Hisata & Yamai also include a model that includes correlations between the price impact and the transaction price. This is particularly useful for the modelling of liquidity risks during times of stress when, as discussed in Chapter 2.5, price-impacts tend to vary negatively with executed prices.

### Model Implementation

Hisata & Yamai [49] implement their model and compare it to conventional VaR. They offer no insight, however, into how they estimate their input parameters.

The results of the comparison indicate that while standard parametric VaR is linear in position size, their L-VaR increases by a factor of about 22 for a tenfold increase in the position. This is good proof that their model captures liquidity costs which are correlated to position size.

Beyond this, they find that if the market-impact increases by a factor of 10 then L-VaR only changes by approximately 2.15. A 25% change in the impact, either way, induces only a 10% change in the L-VaR. This implies that the model is not especially sensitive to errors in the estimation of the market impact.

Loebnitz [65] compare the linear price-impact L-VaR with the stochastic market-impact model. They find that the effect on L-VaR is small and almost negligible and attribute this to the fact that transaction costs are generally dominated by changes in the asset price rather than volatility in the market price-impact.

### Critique

The model presents the most extensive and insightful handling of liquidity risk in VaR. Not only does it cover all of the major components of liquidity risk as discussed in Chapter 2, but it allows for interesting extensions that account for a variety of price-impact functions. All of this is accomplished while retaining the dependency of VaR on trade size.

Although the added complexity has an attendant penalty in that it requires

more in-depth data and parameter estimation, these additions can be ignored in favour of a simpler, more standard formulation.

## 4.6 Model Comparisons

The preceding discourse on the technical details of each measure/model and its strengths and weaknesses ignores the more important qualitative points regarding whether or not each model adequately captures the definition of liquidity risk and secondly whether they embody the preferred characteristics of a “good” measure of liquidity risk.

As noted in Chapter 2, liquidity risk should be defined as the “risk of future loss in value faced . . . due to unforeseen changes in the expected liquidation value” of an asset. Liquidity risk measures/models must, at minimum capture some aspect of this variability. Ideally, however, they should capture this and most of the characteristics noted in Section 3.7, while remaining compact and tractable.

Taking each measure/model separately one can assess them in terms of these general yet important features:

1. **Spread-Based Measures** – As noted earlier, spread-based measures partly measure the cost associated with reversing a small trade. They arise because of information asymmetry and inventory effects which impact on the price formation process. While they do represent one of the most direct costs associated with the uncertainty in liquidation value, they also ignore many of the additional costs which impact on liquidation risk like price risk, price impacts and the opportunity costs of trading.

Spread-based measures are not useful in OTC markets and become less effective in times of stress when price-impact concerns gain greater prominence. They do not measure the variability in actual liquidation value and are inconsistent with much of the micro-structure theory and the elements of a good measure noted in Section 4.4. They also provide no objective gauge of their accuracy and are prone to extreme volatility which can hinder decision-making.

The measures do, however, provide a quick and easy gauge of market depth and provide useful information on the market consensus of liquidity over time.

2. **Trading-Activity Measures** – Volume and Turnover are the most notable measures in this category with each proxying for market-related breadth. Similar to the spread-based measures, while these do provide a market consensus view of liquidity, they tend to ignore many of the costs directly attributable to liquidation. They thus do not conform to the definition of liquidity risk nor do they meet many of the requirements of an ideal measure. They are, however, useful as a quick and rough gauge of liquidity.
3. **Price-Impact Measures** – Price-impact measures capture market-related resiliency and are more internally consistent with the definition than preceding measures.

Measures like the Amihud-Illiquidity Ratio and the P & S measure capture the implicit costs associated with trading but ignore many of the other directly attributable costs like the spread, price risk and opportunity costs. They also provide no practical adjustment for market crises which helps them capture these better.<sup>5</sup>

The Amihud-Illiquidity Ratio and the P & S measure both have methodological drawbacks as both of them make no distinction between temporary and permanent price impacts nor are their price impacts adjusted to the amount considered for trade.

Like the preceding measures, these can also be implemented with readily available data and have been shown to co-ordinate well with known periods of illiquidity. They do seem to capture an aspect of liquidity risk which is different to the other measures.

In general all of the above measures give a relatively good indication of the market consensus of liquidity through time. They are not objectively verifiable, however, nor are they particularly well-adjusted to a specific trade.

#### 4. **Spread-Adjustment Models**

All of the spread-adjustment models require relatively little data and are generally easy to understand. They, however, notoriously ignore quantity-price effects and concentrate solely on the spread.

- **Bangia Model** – The Bangia *et al* model is laudable and easy to understand. With the use of a relatively simple addition to standard VaR, it easily captures some of the costs associated with spread-related variability.

Despite this, the model's central premise that one can completely separate liquidity risk from price risk contradicts the ideas in

---

<sup>5</sup>Of course the measures can be estimated with historical data that reflects a market crises

Chapter 2 which points out that price impacts, etc. directly affect a trader's price risk. The model is thus incomplete and ignores many implicit (and explicit) liquidity-related costs and the price-risk/trade-risk trade-off. Moreover its assumption that extreme events in the spread occur concurrently with extreme return events may make the model more inaccurate in times of stress.

The model does not really come close to capturing the variability in liquidation value brought about by market frictions. Indeed even the way it accounts for spread risk is not via the influence of the spread on end liquidation value. Thus while the model is simple it does not completely account for the definition of liquidity risk. It also provides no gauge of the market consensus of liquidity over time.

- **Ohsawa & Muranaga** – The Ohsawa & Muranaga Monte Carlo L-VaR model seems ad hoc and unsubstantiated by theory. While the model does come close to capturing price-impacts via its  $\lambda$ -adjustment, the addition is not robust and many other liquidity-related transaction costs are ignored. The model also only indirectly incorporates spread-related variability with the use of VWAP, which is not representative of the true spread.

The model only marginally reflects the variability in an asset's liquidation value, has no adjustment for times of stress and is generally inconsistent with market micro-structure theory.

- **François-Heude & Van Wynendale** – The François-Heude & Van Wynendale model overcomes the correlation problem in the Bangia model, while retaining the model's original simplicity. It also extends the original Bangia model to account for volume-related effects with the use of a quantity-adjusted spread. In this way the model accounts for the variability in the spread and, to some extent, endogenous risk as well.

The model, however, ignores true price impacts, temporary vs permanent price effects, a  $\tau$ -penalty trade-off and explicit costs of trade. The quantity-adjusted spread is also difficult to implement as it requires data like the 5 best order book limits which is not readily available.

In general the François-Heude model, like the Bangia framework, only indirectly reflects the market consensus of liquidity as it is based on a theoretical bid price. It also only captures depth and ignores breadth and resiliency.

- **Angelidis & Benos** – The model is basically the Bangia model applied to a structural inventory model which mimics the true spread. Thus while it incorporates endogenous risk through the

modelling of volume-effects on the trade price, it suffers from many of the same shortcomings as the Bangia model. The use of the structural inventory framework makes the resulting L-VaR consistent, but leaves the loss forecasts more open to model-error and estimation error, as if the underlying inventory model proves to be inadequate then the entire L-VaR framework fails. The structural model does, however, allow the model to capture breadth via the parameter adjustments.

Like the preceding models, quantity impacts which are not expressed through the spread (like temporary vs permanent effects, etc.) are ignored. The  $\tau$ -penalty is also ignored and there are no model adjustments for stress. The model is silent on many of the implicit liquidity-related transaction costs and provides no view on market-wide liquidity. Most significantly many of the important spread-related aspects like spread volatility and its absolute level are also left out.

The model does seem to be indicative of the market micro structure and is both easy to understand and implement.

## 5. Parametric-Adjustment Models

All of the spread-adjustment models and the models which follow are based on a parametric VaR framework. They thus suffer from the same non-liquidity-related drawbacks as standard parametric VaR.

- **Al Janabi Model** – The model is detached from micro-structure theory and the price formation process. It offers very little more than standard parametric VaR and in no way captures the definition of liquidity risk. While the time-scaling adjustment it offers is easy to implement, it is based on questionable underlying assumptions.
- **Shamroukh Model** – The model has the same drawbacks as the Al Janabi model as it focuses on a scaling-adjustment to account for liquidity risk. The Shamroukh model, however, offers a better development of this adjustment which is more grounded in the liquidation process.

Although the model's underlying assumptions are less questionable than the Al Janabi model, it still assumes that the price process is independent of the liquidation process. Moreover its price impact formulation is simplified and very similar to the P & S Measure, attributing the entire volume-related return-reversal to the impact of trade. Like many of the models, it assumes linear trade which is not really representative of true trade. The model in no way accounts for the variability in the spread, does

not model price risk and has no adjustments for times of stress. It can, however, be easily implemented with readily available data.

- **Berkowitz Model** – Unlike other L-VaR models, the Berkowitz Price-Elasticity model requires portfolio cash flow and trade price data in order to be implemented. Such data is only readily available if the practitioner maintains a record of internal trades and flows and is difficult to source by external parties.

Generally the portfolio version of the model is impractical as it requires numerical modelling to implement. Moreover the model is based on a structural price-change model and thus shares some of the drawbacks of the Angelidis & Benos model.

Although the spread is not specifically modelled, the majority of costs associated with price impacts and many of the other aspects of liquidity, are accounted for to the extent that these costs are present in a portfolio's realized cash flows. Thus while the model is not completely consistent with theory or the definition, it does offer a black-box view of the impact of liquidity with a much lower modelling burden.

## 6. Optimal Liquidation Strategy Models

OLS models are more complete than the other models. However, they require much more data and onerous estimation procedures.

- **Jarrow & Subramanian Model** – The model is certainly internally consistent as it measures a wide spectrum of liquidity-related costs. While it ignores explicit costs, these can easily be modelled by a higher stochastic discount.

Since the model is based on a GBM (like many of the OLS models) there are questions regarding how well it reflects the market consensus of liquidity through time. This may prejudice the model during times of stress as the parameters involved in a GBM require time to reflect regime changes.

In addition to this the model ignores temporary price impacts, spread-risk and lacks a  $\tau$ -penalty function. It also ignores depth and breadth and focuses solely on resiliency whose impact, via the quantity discount parameter, seems difficult to estimate.

- **Almgren & Chriss Model** – The Almgren & Chriss model is basically a parametric VaR in which the input mean and variance are constructed to minimise the costs associated with a specific trading strategy.

While accurate and robust, the model requires parameter inputs which are difficult to estimate. Indeed the formulation offers no



insights as to how some of the inputs, like  $\eta$  should be estimated. Data requirements are also relatively onerous.

The model does offer a rich price-impact formulation accounting for temporary and permanent price impacts. It, however, accounts for the price-risk/liquidity-risk trade-off only indirectly via the optimal liquidation strategy which is derived to limit costs. There is no specific  $\tau$  trade-off as  $\tau$  is assumed to be constant and exogenous. In addition the model only offers a closed-form, practicable solution if price-impacts are assumed to be linear and deterministic – this is unrealistic.

In addition much of the variability in the spread is ignored as only the spread level is accounted for via the “fixed cost” parameter,  $\xi$ . Unlike the Bangia model depth is only partially explored and breadth is generally ignored.

The model does go a long way in capturing the notion that liquidity risk arises from the variability in the expected proceeds from liquidation. This comes at the price of a market-wide consensus view of liquidity and adjustments for times of stress.

- **Hisata & Yamai Model** – The Hisata & Yamai formulation is largely similar to the preceding model. However since  $\tau$  is endogenous here, the model more carefully considers the price-risk trade-off.

The fact that  $T$  is selected so that the liquidity cost function is minimized also greatly simplifies the optimization and, in the continuous case at least, provides much greater flexibility in specifying more realistic price-impact formulations. As shown, in this framework price-impacts can include stochastic coefficients which are correlated to asset returns. Such alterations greatly influence the model’s accuracy, especially in times of stress when correlations become more marked.

In general, however, such added accuracy carries with it additional data burdens and the model becomes cumbersome as it becomes more complete and realistic.

Once again the model ignores the variability in the spread.

The general impression gleaned from the above comparison is that models which tend to accurately account for the spread and its variability tend to do so at the expense of accurate price-impact modelling, while models which offer a rich quantity-impact framework tend to ignore the risks posed by the spread. As noted in Chapter 2, a complete liquidity risk model must account for the variability in the spread, the absolute level of the spread, price impacts, price risk and model the trade-off between price risk

and trade-risk. Almost none of the models completely capture all of these aspects and very few offer adjustments which make their estimates of loss more conservative during times of stress.

While the L-VaR models generally come closer to the requirements of Section 3.6 as they offer objectively verifiable estimates and are ostensibly more complete, they also tend to perform badly in describing the market consensus of liquidity over time. Contrastingly, while the measures offer important market-wide information pertaining to particular aspects of liquidity, they do a bad job of integrating all the effects into one impact.

None of the models or measures come close to completely capturing the needs of a liquidity risk measure, as through the constraints of modelling, they inevitably ignore some aspect of liquidity risk in favour of another. The implication is that liquidity risk can only be measured with an array of measures. The Bangia model requires a price-impact measure like the P & S Measure to capture resiliency. Alternatively the OLS models need to capture breadth and depth and thus must be monitored in conjunction with the spread and volume traded. The general idea is that a model which requires much data modelling should be enriched with a measure/model that is quick and easy to implement as the data requirements should not be too onerous so as to make overall implementation intractable.

Given the need to consider data requirements, Tables 4.2 and 4.3 summarise the Inputs, Outputs and parameters associated with each model. TS denotes “time series” and VaR, the standard parametric VaR defined in Chapter 3. As can be seen each of the models, require a range of differing inputs which can be difficult to acquire and makes implementing a wide range of models simultaneously more difficult.

Model/Measure	Output	Parameters	Inputs
Bid-Ask Spread	$A_t - B_t$	N/A	TS of Bids & Asks
Traded Volume	$V_t$	N/A	TS of Traded Volume
Turnover	$\frac{V_t}{\text{Shares Outstanding}}$	N/A	TS of Traded Volume & Shares outstanding
Amihud Ratio	$\frac{1}{D_t} \sum_{d=1}^{D_{t,i}} \frac{ R_{t,d}^i }{V_{t,d}^i}$	N/A	Daily stock returns $R_{t,d}^i$ & volumes $V_{t,d}^i$
P & S Measure	$\zeta_{i,t}$	$\zeta_{i,t}$ OLS Regression coefficient	Stock(i), market(m) returns & traded volumes: $r_{i,d,t}, r_{m,d,t}$ & $V_{i,d,t}$
Bangia Model	$LVaR_\alpha = P_t[1 - \exp(\mu_t - \alpha\theta\sigma_t)] + \frac{1}{2}P_t(\bar{S} + \tilde{\alpha}\tilde{\sigma})$	$\mu_t, \theta, \kappa,$ $\sigma_t, \bar{S}$ & $\tilde{\sigma}$	TS of Mid-prices $P_t$ & Relative spreads $S_t$
Adjusted Bangia	$LVaR_\alpha = P_t[1 - \exp(\mu_t - \alpha\theta\sigma_t)] + \frac{1}{2}P_t e^{\mu_t - \alpha\theta\sigma_t}(\bar{S} + \tilde{\alpha}\tilde{\sigma})$	$\mu_t, \theta, \kappa,$ $\sigma_t, \bar{S}$ & $\tilde{\sigma}$	TS of Mid-prices $P_t$ & Relative spreads $S_t$
Ohsawa-Muranaga 1	$P_{Ex} = P_{VWAP}^0 \exp(\sigma_{VWAP} \epsilon_a \sqrt{t}) + \sigma_H \epsilon_b$	$\sigma_H$	TS of VWAP, TS of daily prices & Random std normal variates
Ohsawa-Muranaga 2	$P_{Bid} = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) - \frac{1}{2}f(u)$ $P_{Ask} = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) + \frac{1}{2}f(u)$	$\sigma_H$ $\sigma_M$	TS of VWAP, TS of daily prices & Random std normal & uniform variates
Ohsawa-Muranaga 3	$P_{Bid}^i = P_{Bid}^{i-1} \exp(\frac{-V_{g\lambda}^i(u)}{NMS})$ $P_{Ask}^i = P_{Ask}^{i-1} \exp(\frac{V_{g\lambda}^i(u)}{NMS})$ $P_{Bid}^0 = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) - \frac{1}{2}\mathbb{E}((f(u)))$ $P_{Ask}^0 = P_M^0 \exp(\sigma_M \epsilon \sqrt{t}) + \frac{1}{2}\mathbb{E}((f(u)))$	$\sigma_H$ $\sigma_M$ $\lambda$	TS of VWAP, TS of daily prices & Random std normal & uniform variates TS of trade prices & sizes
Heude-Wynendaele	$LVaR_t = Mid_{BL_t}[(1 - (1 - \frac{S_{pBL}}{2}) \cdot e^{-\alpha\sigma}) + \frac{1}{2}(S_{pBL,t} - S_{pBL})]$	$\sigma, \tilde{\sigma}$ & $\bar{S}$	TS of Mid-prices, bids and asks
Angelidis & Benos	$LVaR = VaR + [(\theta + \kappa)\sqrt{V_t^{\alpha'}} + \phi]$	$\theta, \kappa,$ $\sigma, \mu$ & $\phi$	TS of intra-day prices, sizes & signs

Table 4.2: Model Data Requirements 1

Model/Measure	Output	Parameters	Inputs
Al Janabi Model	$LVaR = VaR \sqrt{\frac{(2t+1)(t+1)}{6t}}$	N/A	Time to liquidation, t
Shamroukh Model	$LVaR = \sqrt{Q'WQ} = \left(\frac{(n+1)(2n+1)}{6n^2}\right)^{\frac{1}{2}} (VaR)$	N/A	n, the number of trade dates
Berkowitz Model	VaR based on distribution of $y_{t+1} = Q'_t(p_t + x_{t+1} - \theta q_t^*)$	$\theta$	Historical portfolio values $p_t$ , net flows $q_t$ & trade amounts $x_t$
Jarrow-Subramanian Model	$LVaR = ps \mid [\alpha - \frac{\sigma^2}{2}] \mathbb{E}[\Delta(s)] + \mathbb{E}[\ln(c(s))] - 2[\sigma \sqrt{\mathbb{E}[\Delta(s)]} + \mid \alpha - \frac{\sigma^2}{2} \mid std[\Delta(s)] + std[\ln(c(s))]] \mid$	$\mathbb{E}[\Delta(s)], \mathbb{E}[\ln(c(s))]$ $\sigma, std[\Delta(s)] + std[\ln(c(s))]$ Estimation easier if $c(s_i)$ & $\Delta(s_i)$ deterministic	TS of portfolio trading times & historical trade prices TS of Bids, Asks &
Almgren-Chriss Model	$LVaR_p = \alpha \sqrt{Var[x]} + \mathbb{E}[x]$ $\mathbb{E}[x] = \frac{1}{2} \gamma x^2 + \epsilon \sum_{k=1}^N \mid \eta_k \mid + \frac{\eta}{\tau} \sum_{k=1}^N \eta_k^2$ $Var[x] = \sigma^2 \sum_{k=1}^N \tau x_k^2$	$\eta, \gamma, \kappa$ Estimated by fitting linear case	TS of high-frequency trade prices, order sizes & quotes. Position size
Hisata-Yamai Model	$LVaR = \left(\frac{2\eta\sigma^2 Z^2 X^2}{3r}\right)^{\frac{1}{3}}$	$\eta, \sigma$	As per Almgren-Chriss

Table 4.3: Model Data Requirements 2

## Chapter 5

# Model Implementation and Analysis

The theoretical comparisons of the measures and the earlier chapters on the constitution of liquidity risk make it clear that the full extent of liquidity risk cannot be readily captured by any single measure or model. The most accurate and complete view of liquidity is brought about by investigating a set of different measures/models simultaneously. This contention arises naturally from the theoretical discussion.

Although the theoretical discussion highlights many of the models' qualitative strengths and weaknesses, an empirical analysis is still required to determine which of the set of measures/models is best suited to the management of liquidity risk. The models' accuracy and practicability needs to be tested to determine whether they are useful or not.

The difficulty lies in designing an approach which tests how well the full spectrum of measures/models perform at capturing the totality of liquidity risk.

### 5.1 Data & Methodology

The analysis and testing of the L-VaR models is made easy by the fact that VaR models make objectively verifiable estimates of future loss. As a consequence VaR models can easily be compared to realized losses and back-

tested to verify their accuracy. Indeed a number of statistical tests (which will be discussed later) have been designed to validate VaR-type models.

Such tests determine the accuracy of the VaR model's forecast of future loss relative to realized portfolio losses. They do not necessarily determine what aspect of liquidity risk the models capture. However, to the extent that liquidity-related costs (as noted in Section 3.5) impact on portfolio loss they do test how accurately the models account for liquidity-related costs.

The measures, however, do not lend themselves to objective testing. In order to assess them, the literature, as noted in Section 4.4, has taken the general approach of collating different measures over time and then comparing them by studying their overall time series patterns and characteristics. This is done with a view to assessing how well their behaviour correlates with periods of known illiquidity. In this way one can assess whether the measures actually capture information on liquidity at all.

In addition to this, a few papers have studied the correlations between different measures and have undertaken a principle component analyses. The goal of both analyses is to determine whether different measures capture different aspects of liquidity risk. Although the idea behind the analysis is sound, the implementation in the literature has been done rather blindly with no regard being given to the possible non-stationarity of the data and the potential for spurious correlations. The analysis which follows corrects for this oversight, but still suffers from the same problems of subjectivity as earlier studies.

The measures, being descriptive, are inherently subjective and thus cannot be properly tested to determine how well they capture liquidity information. At best all that can be said with current techniques is that a specific measure seems to capture a specific aspect of liquidity and is sensitive to liquidity events.

Given this disparity between the L-VaR models and the liquidity measures, two analyses are conducted: one for the measures and another one for the VaR models. All the measures and models are, however, computed from data on the same stocks.

In order to highlight the differences in the measures/models response to illiquidity, data on 4 different stocks has been gathered. The chosen stocks represent 4 differing degrees of liquidity from highly liquid to illiquid. So that the analysis is not artificially biased towards one measure or another, no objective liquidity measure was used to rank the stocks. Rather the

insights of experienced equity traders into market liquidity guided the selection of Anglo American Plc (ANG), Pick n Pay Holdings (PIK), Coronation Management Company (CML) and Seakay Ltd (SKY). In this way the intuitive, well-known aspects of liquidity as experienced by market-practitioners is evident in the selection.

Of the 4 stocks ANG and PIK represent shares which are generally regarded as more liquid than the other pair, with ANG being more liquid than PIK. CML and SKY represent illiquid shares with SKY, a newer listing with a smaller capitalization, being the most illiquid share of the selection. All of these stocks are listed on the Johannesburg Stock Exchange's (JSE) main board.

### 5.1.1 The JSE

The JSE is the world's 16th largest stock exchange [13] and contains a wide diversity of liquid and illiquid stocks. Although it is a fully-electronic, order-based exchange and thus fairly liquid overall, the market remains far from frictionless. Its operations are subject to the same discontinuities as other exchanges.

All trades on the exchange go through the standard LOB market operations (as discussed in Section 2.1) of order routing, execution and clearing and settlement. Order routing occurs via a broker who is a regulated "member" of the exchange while capture and execution occur via JET (Johannesburg Equities Trading). All executed trades are settled via STRATE (Share Transactions Totally Electronic). Orders can take on any number of conditions like fill or kill, execute or eliminate, good till time, good till auction or good till cancelled [12].

Trade occurs between 08:30 and 18:00 each working day, with opening and closing auction periods which precede and end trading. The auctions ensure that prices in the morning and the evening are not too volatile in response to information which has either been gathered towards the close of the day or overnight on the previous day. This facilitates fair price formation and encourages trade but breaches the requirement of a frictionless market that all trades be executed at one perfectly competitive price.

Like other LOB markets, the entire order book's depth is visible [12] to ensure transparency and liquidity. Since there are no market-makers making firm quotes, however, trade is never guaranteed and many stocks can go a full day without trading. Orders in the order-book are matched during

automated trading on the basis of time and price, while matches are made in the auctions based on an algorithm. Unlike the frictionless environment wherein any order size can be traded, the JSE maintains a minimum of 1 share or a multiple of the Normal Market Size (NMS), depending on the security being traded. All orders also have maximum sizes which vary according to the characteristics of the stock [12].

Usually trading is continuous but can be stopped by the exchange if the JSE's trustees deem that a fair price formation process cannot be guaranteed or if additional liquidity needs to be gathered in a particular instrument [12]. Trade has historically only stopped completely due to technical errors but many illiquid securities have their automated trading interrupted by auctions at regular intervals. Volatility auctions may also take place if a security's price moves beyond a certain limit. These are used to prevent abnormal spikes in prices.

Although every effort is made by the exchange's systems to ensure smooth and efficient price formation, the totality of the above trade restrictions render prices liable to discrete jumps. These tend to occur around auctions and during trade halts, when order flow is restricted. The effect of this is that liquidity concerns become especially important during these times.

As with all large international markets, traders face standard liquidity issues like the Bid-Ask spread, price risk (which vary with the order conditions), opportunity costs and price impacts. Price impacts are a particular problem with smaller counters.

Overall while the market is well-known for being resilient (in that it adjusts to order imbalances quickly and prices new information fast), the fact that it is highly concentrated renders traders open to significant liquidity risks. Indeed the top 5 counters account for nearly 65% of the total exchange's market capitalization. Moreover, as highlighted in Section 3.2, the fact that the exchange is fairly open to foreign investors and integrated with overseas markets (particularly the London Stock Exchange) in terms of access to data and technology, means that price impacts change quickly and often and that liquidity becomes a critical issue in times of stress.

As noted in Section 3.4, models which ignore liquidity costs and market frictions can be prone to large errors. Risk models applied to the JSE, of necessity, must then account for the frictions that beset it.



### 5.1.2 Data

As a sophisticated and fully-electronic order-based exchange, transaction data on the JSE is, within limits, not entirely difficult to find. Generally three large data providers dominate the financial services industry: INET Bridge, Reuters and Bloomberg. All three are widely used and trusted by market practitioners with the provision of accurate historical and real-time data on a range of market variables.

#### Measures Data

Given that the measures are mainly analysed by studying their time trends, an effort was made to obtain the longest time series of data possible. Since many of the measures, like the Amihud ratio and the Pastor & Stambaugh measure, use daily data in their computation and since there is a difficulty in finding meaningfully long-term financial data at frequencies higher than a day, daily data was preferred.

Daily data was sourced directly from INET Bridge over the period April 1994 to September 2009. This covers the bulk of the post-apartheid economy and captures the major financial events in the JSE's history. Where a stock like SKY or CML did not exist as far back as that, data only covering their history was extracted.

All data excludes non-trade days, mis-pricing and data errors. Moreover where prices could not be sourced for successive days (due to data storage issues for example back in 1994), the series was treated as a contiguous data set. This had no effect on the analysis as all comparable data suffered from the same problem on the same day.

For the measures daily data on the following market variables was sourced for each stock: the quoted market close price, the quoted daily bid and offer<sup>1</sup>, the daily trade volume and value, the market capitalization and the return of the FTSE/JSE All Share index (the market index).

The above market variables were then used to create the following measures: the Bid-Ask spread, the traded Volume, Turnover and their scaled counterparts (scaled by market capitalization), the Amihud ratio and the P & S

---

<sup>1</sup>The day's close, bid and offer prices represent the prices quoted for the last trade of that day prior to the closing auction, these are used widely in South Africa for modelling purposes as most portfolios are valued using closing prices.

measure. The scaled volume and turnover variables were used to prevent the effect of an escalating company size over time from biasing the results. Since both the Amihud ratio and the P & S measure use daily data to create a monthly measure, all the variables were aggregated over each month to create a monthly time series for comparability.

## **L-VAR Models Data**

The selection of data for the estimation and testing of the L-VAR models was beset by two problems: firstly acquiring a rich enough frequency so that all the models could be simultaneously tested and accurately implemented and secondly ensuring that the data set covered a period long enough to ensure that the results are not weak.

The implementation of the models require the estimation of parameters which are not directly observable and depend crucially on the market micro-structure. For this reason high-frequency tick data was needed as daily or other data would mask the micro-structure effects of price-impacts, etc. The problem is choosing a frequency that mitigated the difficulties mentioned earlier.

Generally tick data is extremely difficult to source over long time periods as market practitioners do not commonly make use of it. Moreover at extremely high frequencies (say every 1 minute or so), variables may not be changeable enough to facilitate proper parameter estimation – for instance if a stock only trades every 5 minutes, which is common for the illiquid stocks, then its bid-ask spread will be necessarily zero for much of the estimation period. This would invalidate its usefulness as a liquidity variable.

In order to balance these needs, half-hour data for a period as long as could be reasonably sourced was used for all 4 stocks. A data set of 366 trade points, covering the period 6 May 2009 to 8 September 2009 (125 days) was acquired from Bloombergs, comprising the following variables:  $T_p$ , the trade price, being the closest preceding transaction price, and its associated volume,  $T_v$  and the ask price  $A_p$ , ask volume  $A_v$ , bid price,  $B_p$  and associated bid volume,  $B_v$ .

### 5.1.3 Methodology

#### Measures

The analysis of the measures begins in much the same as way as previous papers. Firstly an analysis is conducted into the significant periods of known illiquidity on the JSE. The measures are then analysed against this to test how well they hint at such events and whether their behaviour is synchronised with the periods of illiquidity. This helps shed light on whether the measures capture liquidity information at all.

Secondly the correlations between the measures are examined. The view put forward in the literature is that if two measures are highly positively correlated then it is likely that they capture the same aspect of liquidity risk. Unfortunately with time series data such inferences are not completely water-tight as financial time series are often non-stationary, meaning that the correlation coefficient between two variables could merely be measuring the mutual time drift between them and not the co-variation based on similar reactions to new information.

Moreover the presence of non-stationarity implies that both standard correlation analysis and the bulk of time series modelling techniques like spectral analysis, etc. cannot be used. This complicates the analysis of multiple time series. Generally the literature argues in favour of transforming the original time series (by differencing or other techniques) until it is stationary and then conducting an analysis on this transformed series. Alternatively tests of co-integration can be used to examine the long-term relationship between non-stationary variables.

In order to correct for the problem of spurious correlation in the analysis, all the time series were tested for stationarity using the standard Dickey-Fuller unit root test in *Eviews*. The test was conducted at various lags of autocorrelation and included a constant and drift term. At least in this way the analysis can be appropriately qualified. Non-stationary time series were also appropriately differenced until they were stationary, the resultant differenced series were then subjected to a further correlation analysis.

A co-integration approach was not favoured as tests for co-integration merely indicate whether two series are related in some long-term relationship, not whether they co-move as suggested when two variables capture the same information. Co-integration reveals long-term trend relationships rather than proxy information.

Finally in order to add some concreteness to the results of the correlation analysis, the measures were subjected to a principle component factor analysis. Factor analysis is a statistical analysis technique that tries to explain the observed variation in a set of variables by extracting factors that account for the linear-relationship between the variables. Factor analysis attempts at explaining the co-variation between variables as a function of the factors with which they are highly correlated. In this way it helps determine the extent to which different variables capture similar information.

The underlying assumptions of factor analysis are much less rigid than that of correlation or time series analysis. Factor analysis only requires limited homoscedasticity and stationarity assumptions and this is not essential. Moreover variables need not follow any pre-specified distribution for the results to be significant, although multi-variate normality is required for hypothesis testing. The crucial assumption is that the variables are not independent but bear at least some linear relationship to one another. Mild multi-collinearity also does not bias the results but can raise the standard errors of the factor loadings, making their interpretation difficult. Importantly factor interpretation must be based on theory and cannot be subjectively inferred from the data set itself. All of the above makes factor analysis ideal for the task of uncovering to what extent different liquidity variables capture similar information on different aspects of liquidity.

Factor analysis was conducted in *Statistica* using the principle components methodology for extracting variance. With this technique a factor is extracted based on its ability to explain the total variation in the original data set. Successive factors are then extracted based on their ability to explain the remaining variability in the data set. In this way successive factors are uncorrelated, making their interpretation easier. Varimax rotation was also undertaken to facilitate interpretation.

## **VaR Models**

The L-VaR models are tested by comparing the loss forecasts of each model against one another and against standard, well-known VaR regimes. In order to add objectivity to the model comparisons, however, all of the models were backtested against the realized losses of a simulated trading strategy which specifically accounts for liquidity-related costs. This is crucially different to previous papers which merely compared the modelling approaches and did not test forecast accuracy.

Using the raw half-hour trade variables sourced from Bloombergs, 13 dif-

ferent, 99% Value-at-Risk models were implemented over the period 7 May 2009 to 8 September 2009. A 99% confidence level was chosen so as to highlight the effects of ignoring liquidity-related risks at extreme levels of loss. As argued earlier, at these extreme levels of market stress liquidity plays a more pivotal role.

The rolling forecast of future loss across the different VaR models was back-tested against profits and losses from a simulated trading strategy involving the delta hedging of a portfolio that is long a position in the relevant share, short a vanilla call option on the same and long an amount of cash. Although half-hour profit and losses from realized trading strategies would be the most ideal for backtesting, these are extremely difficult to acquire for two reasons. Firstly in reality market-practitioners do not value their portfolios every half-hour. Secondly in order to limit transaction costs, practitioners do not trade every half-hour, nor do they aim for delta neutrality every half-hour. Traders prefer to re-balance their portfolios on delta daily and to hedge the gamma of the position for further protection. This means that trades only generally occur a few times a day.

Delta-hedging, however, does provide a natural and easy-to-understand sequence of trades upon which one can test VaR models. The hedging strategy is a very common exercise amongst traders and frequent hedging has imminent liquidity consequences, even if these are only in the form of exogenous trading costs. Moreover testing how well VaR models perform in forecasting the loss arising from such a trading strategy is interesting in its own right. Assessing how well incorporating liquidity-related costs into this forecast and, by extension, understanding to what extent liquidity influences the ability to hedge provides further interesting information.

The simulated trading strategy is constructed by assuming that a trader holds 6,000 units of a particular share, be it ANG, PIK, CML or SKY at time  $t_0$  which corresponds to 09:00 on 6 May 2009. At this time the trader is short  $6000/\Delta^2$  call options, all with the same strike price, a time to expiry of 0.34 years and a volatility of 20%. The strike price is set differently for each share so as to ensure that the options have non-zero value at  $t_0$ . The option contracts are assumed to be written on one share. Thus the net effect of being long 6,000 shares and short  $6000/\Delta$  call options is that if the share price falls by  $x\%$  then  $6000x/100$  would be the loss in the share position. This would be exactly offset by the increase in the option value of  $(60000/\Delta)\Delta x\%$ .

In this way as the stock price changes (based on the half-hour transaction

---

<sup>2</sup> $\Delta$  here denotes the delta of a vanilla call option as calculated by Black-Scholes

price) so does the delta of the option, meaning that the trader must re-balance his position to restore delta neutrality. Since the strategy aims at simulating actual trading conditions, one cannot assume an unlimited trade size at each hour, nor can one assume that the trader who aims at delta-neutrality can trade unlimitedly at the same price. These are the fundamental assumptions which undermine VaR. In order to account for this, the following rules are included in the trading strategy: a trade “build-up” rule and a “price impact” rule.

The trade “build-up” rule operates as follows: if the amount required for trade at the  $i$ th time step,  $Q_i$ , is positive and there has been no trade executed at that half-hour or offered, so  $T_p = 0$  and  $A_p = 0$  then no trade occurs and the amount which would have been traded will be rolled forward to  $t_{i+1}$ , at which time both the “trade backlog” from the previous time step and the amount necessary to restore Delta neutrality is traded. The same occurs if the amount required for trade is negative and both  $T_p = 0$  and  $B_p = 0$ . This is done to enforce the idea that trades build-up in times of illiquidity making further trading more difficult and more costly.

The “price impact” rule exists so as to counter the unrealistic assumption that trade can occur without moving the price. It captures the endogenous component of liquidity risk and operates as follows: in a buy transaction, that is  $Q_i > 0$ , if the amount to be bought is lower than  $T_v$  then it is assumed that the entire transaction goes through at the associated price  $T_p$ . This makes the cost of the purchase  $Q_i * T_p$ . If, however, the opposite is true than, we assume that part of the transaction, namely  $Q_i - T_v$  is traded at  $T_p$ , with the balance traded at gradually worsening prices. First if the balance can be captured by the ask size  $A_v$  then it is assumed that trade occurs at  $A_p$ . If, however, the combined amount  $A_v + T_v$  is still smaller than  $Q_i$  then it is assumed that the remainder is traded at  $A_p * k$ , where  $k$  is a penalty price-impact factor which increases in the remaining trade size. A similar procedure occurs for sales. Figure 5.1 illustrates the trading strategy.

Finally in order to account for the exogenous costs associated with trading in South Africa, all trades are charged at the standard rates charged by publicly available brokerages. The brokerage costs go through the portfolios cash account and thus stock need not be sold to meet it. The costs include a brokerage equal to 0.4% of the transaction value subject to a minimum of R120, STRATE costs equal to 0.05% of the value traded with a minimum charge of R10.92 for transactions up to R200 000 and a maximum charge of R54.59 for transactions exceeding R1m. An investor protection levy (IPL) is also incurred at 0.0002% of the value traded, VAT is charged at 14% on the sum of the brokerage, IPL and STRATE and a securities tax of 0.25%

### Trading Strategy Algorithm

**Variables:**  $T_{pi}, T_{vi}$   $\rightarrow$  ith Transaction price and associated volume

$A_{pi}, A_{vi}$   $\rightarrow$  ith Ask price and associated volume

$B_{pi}, B_{vi}$   $\rightarrow$  ith Bid price and associated volume

$Q_i$   $\rightarrow$  ith units traded

$Q_i$  = Units required for Delta neutrality + Backlog

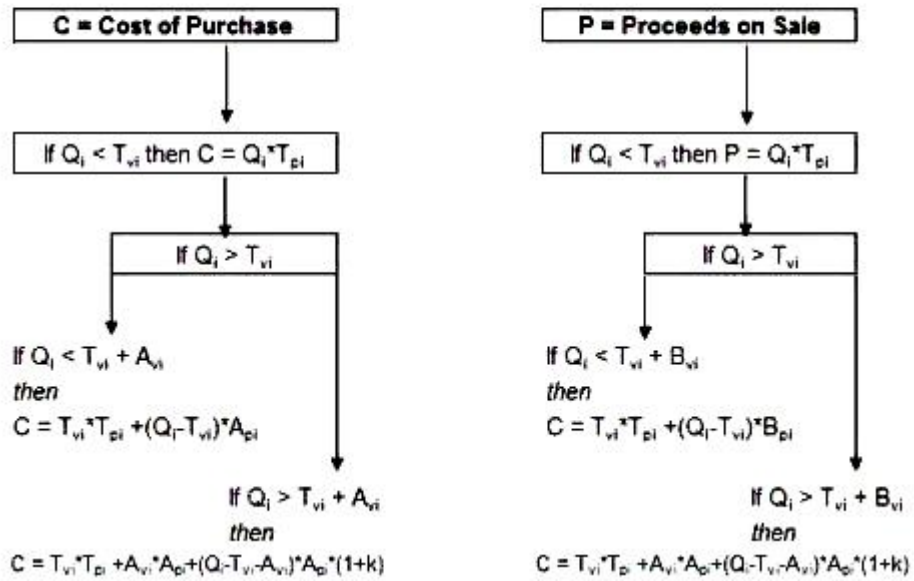


Figure 5.1: Trading Strategy Algorithm

<b>Brokerage Costs</b>	
Brokerage	0.40%
Securities Transfer Tax - purchases only	0.25%
Investor Protection Levy	0.0002%
STRATE Settlement Costs	0.005459%
Minimum	10.92
Maximum if deal > R1m	54.59
VAT - on above only	14%

Table 5.1: Brokerage Costs

of all purchases is also deducted. The details are shown in Table 5.1.

The aim of the trading strategy is to create a series of realized profits and losses for the backtesting of the forecasted VaR numbers. Although backtesting of the VaR models has not been undertaken in the literature, it should be remembered that a simulated trading strategy has its drawbacks and that the first prize for such testing is realized trade data. The problems with the simulated strategy are two-fold.

Firstly estimating the potential price impact when there is insufficient market depth at a particular time is not foolproof. The approach taken here is that amounts in excess of that bid or offered are traded at prices whose discounts are proportional to the excess traded. Since this discount is estimated from the data at hand and thus moves, it is unlikely that the resultant profits/losses will reflect the experience of traders exactly. All that can be hoped for is a reasonable margin of error.

Secondly, the trading strategy is based on half-hourly delta-hedging. Since at least half of the stocks considered are illiquid,  $T_p$ , the price upon which the portfolio's delta is based, does not exist at every half-hour. This is since the stocks do not trade at every half-hour. The effect is that for these illiquid stocks, delta stays the same, meaning that the amount required for trade at each time step is necessarily less. Since the mark-to-market price hardly moves, the risk of the portfolio being delta non-neutral for a particular price change is minimal. The impact is that the illiquid stocks tend to trade less and in smaller quantities than the more liquid stocks. This makes the effect of rampant illiquidity on VaR accuracy less transparent.

Fortunately the flaws of the strategy are mimicked across all the stocks, meaning that the trading strategy is not a control variable. Any discrepancies between the VaR models must thus be accounted for by the differences in the models themselves and not necessarily by the strategy. The drawback



is that without realized data, the analysis cannot be used to account for the behaviour of the models in real-life application.

In total 13 different VaR models – including 8 different L-VaR models are applied to the half-hour tick data and used to calculate a 99% half-hour forecast of future loss based solely on the long stock position. The VaR models applied and the method of application are discussed briefly below:

### 1. Parametric VaR

A standard parametric VaR, based on Normal distribution assumptions, as in Section 3.11, is calculated as a benchmark for comparing the other VaR models. This is the most widely used formulation of VaR and thus showcasing it is important. The inputs to 3.11 are estimated with maximum likelihood methods based on half-hour returns.

In order to highlight the divergence in loss estimates when VaR is based on different mark-to-market regimes, a parametric VaR is calculated based on  $T_p$ ,  $A_p$ ,  $B_p$  and the mid-price,  $M_p$ . These are in turn compared to realized portfolio profits and losses when the portfolio is marked-to-market at each of the trade, ask, bid or mid-prices.

The parametric VaR is deliberately not based on historical portfolio returns but rather pure price returns so as to highlight its differences with the Historical VaR. Using portfolio returns as an input remains a valid modelling choice, however.

### 2. Parametric EWMA VaR

A parametric VaR is applied wherein the estimate of return volatility is based on an EWMA regime with  $\lambda = 94\%$ . This figure was selected as it is the choice of *RiskMetrics*<sup>TM</sup> when they initially published the methodology [73].

Although the parametric VaR also uses a rolling estimate of  $\mu$  and  $\sigma$  as an input, the estimate for  $\sigma$  does not update in a way that down-weights old information relative to new.

### 3. Parametric GARCH VaR

Similar to the EWMA VaR, a GARCH(1,1) methodology was applied to adjust the estimate of  $\sigma$  in the parametric VaR. The parameters were arbitrarily set at  $\alpha = 40\%$ ,  $\beta = 50\%$  and  $\gamma = 1 - \alpha - \beta$ .

### 4. Historical VaR

A historical VaR is applied using Excel's built-in Percentile function. The inputs are the confidence level and a rolling return window of realized portfolio returns.

#### **5. Monte Carlo VaR**

In order to complete the analysis of the standard VaR methodologies, a Monte Carlo VaR is calculated. Using 10 000 simulations of returns, a simulated return distribution is created. The VaR is calculated from this simulated distribution using Excel's percentile function. The VBA code for the simulation is presented in Appendix D.

The returns are generated with a Geometric Brownian motion SDE.

#### **6. Bangia L-VaR**

The standard Bangia model, as specified in 4.8, is applied to the data set. Preference was given to this model over the adjusted Bangia model so as to highlight the original intent of the model.

As with the parametric VaR,  $\mu$  and  $\sigma$ , were estimated using a rolling return window based on price returns. The relative spread parameters were estimated in the same way, with the critical value,  $\tilde{\alpha}$ , estimated using Excel's percentile function. This is in keeping with the discussion in Section 4.5.

In order to account for the possibility of fat tails,  $\theta$  was estimated using the empirical kurtosis,  $\kappa$ , and  $\phi = 0.4$ . Although strictly speaking,  $\phi$  must be estimated by regressing the results of the historical VaR against the specification in 4.8, it was felt that using the original estimate provided by Bangia for a 99% VaR would be satisfactory as the VaR estimate seems quite robust to changes in this parameter.

#### **7. Heude and Van Wynendaele L-VaR**

The Heude and Wynendaele model that is not adjusted to the spread is applied in order to arrive at a liquidity-adjusted estimate of loss. The model as specified in 4.18 is applied, without the quantity-adjustment. The quantity-adjustment was ignored as data on the 5 best order book limits is not readily available, making its application intractable.

#### **8. Angelidis and Benos L-VaR**

The Angelidis and Benos L-VaR model is applied by estimating the parameters in 4.20 and 4.21 and then using them as inputs into the L-VaR equation specified in 4.25.

The input set  $(\theta, \kappa, \phi)$  is estimated by regressing the half-hourly price change  $P_t - P_{t-1}$  on traded volume and the trade sign indicator,  $X_t$ .

#### 9. Al Janabi L-VaR

The Al Janabi model is relatively uncomplicated to implement as it essentially involves scaling the standard parametric VaR by a factor that represents the weighted-time to liquidation,  $t$ . The resultant VaR equation is specified in 4.29.

As stated earlier this input is not commonly available as most traders do not generally have a specified time by which they expect all of their holdings to be liquidated. However, in the case of delta-hedging this is made easier as by the time an option expires or is exercised, the corresponding stock holdings need to be sold.

#### 10. Shamroukh L-VaR

The Shamroukh model also merely involves a scaling of the standard parametric VaR equation. The specification in 4.36 is implemented.

#### 11. Berkowitz L-VaR

The Berkowitz model is implemented by finding an estimate for  $\theta$  using the regression form in 4.41. This is then used in the calculation a portfolio mean and variance as specified in 4.42 and 4.43.

In this way the underlying price-change assumption of Bertsimas and Lo is transported directly into a parametric VaR framework. This methodology is much less computationally intensive and arguably just as accurate as forecasting the entire one-step ahead return distribution with the inversion formula for characteristic functions.

#### 12. Almgren & Chriss L-VaR

The Almgren and Chriss model is implemented by assuming linear permanent and temporary price impact functions, as specified in 4.57. This requires the estimation of the following parameters:  $(\gamma, \eta, \varepsilon)$ .

Since  $\varepsilon$  represents the fixed costs associated with trading, it is estimated as the sum of the relative spread and the average historical cost of trade like brokerage, etc.  $\gamma$  and  $\eta$  are however, estimated from the following regression equation:

$$P_t - P_{t-1} = \gamma V_t + \varepsilon \text{sgn}(V_t) + \frac{\eta}{\tau} V_t \quad (5.1)$$

Estimating price impacts, even from a simple regression equation as above is extremely difficult, particularly for illiquid stocks. Generally since illiquid stocks trade infrequently their price and volume series remain zero for a long period, implying that when a trade actually does go through, the impacts associated with it tend to be extremely large. The effect of this is that the expected costs of liquidation in 4.55 are large and the resulting VaR estimate is unrealistic.

The unusually high volatility of price impacts makes it necessary to assess the parameter estimates for reasonability. Estimating these parameters requires a mixture of market experience and an accurate estimation procedure. The fact that the literature provides so little guidance regarding their estimation further complicates this task.

Despite this, the parameter estimates were made more reasonable by analysing the regression results and then replacing unusually large outputs by the average of the preceding series. In this way the series was normalized relative to its history.

### 13. Hisata & Yamai Model

The Hisata & Yamai model, at least in its continuous-time form, provides a much more elegant and easy-to-implement L-VaR formulation than the Almgren & Chriss counterpart. The model is implemented with the same price-impact parameters as the preceding model and the specification in 4.78.

Significantly if  $\eta \approx 0$  then the L-VaR estimate here is also close to 0. This highlights the sensitivity of both the Almgren & Chriss and this formulation to price-impact estimates.

Of the 18 VaR models discussed initially in Chapter 4, only 13 were implemented and while the above list covers the bulk of VaR methodologies, a mention must be made of the VaR models which were not applied.

From the standard VaR models, only the extreme value theory and the Kernel-estimator VaR models were not applied. These were not tested primarily because they are extremely computationally intensive. Both of these methodologies require fairly complicated transformations and have a host of additions and adjustments. They are a separate research avenue in their own right. Although it would be interesting to determine if adjusting for extreme tail-events eliminates the need for liquidity adjustments, the theory and application of these models falls beyond the scope of this thesis.

Only the Ohsawa-Muranaga VaR model and the Jarrow-Subramanian models were not implemented from the set of L-VAR models. The Ohsawa-

Muranaga formulation is ignored, as the model has no robust theoretical basis for its distributional adjustments. Its adjustments seem ad hoc and inconsistent with the development put forth in other papers. Moreover, as the model depends on a Monte Carlo simulation of returns, and since such a VaR is already implemented, its implementation seemed redundant as it is doubtful that the Ohsawa model would yield interesting results. The same holds for the Jarrow-Subramanian model, since the other optimal liquidation strategy models are more robust and extensive than it; its implementation seemed unnecessary.

Although for completeness, all the models discussed should be implemented, the goal of this thesis is to discuss and test the practical management of liquidity risk. If a model thus fails to be easy-to-implement or seems to mimic another model, then there seems little sense in implementing it.

## 5.2 Results

### 5.2.1 Measures Analysis

As specified the measures are analysed primarily by investigating how sensitive they are to known periods of market-wide illiquidity. In this regard, their analysis begins with a brief investigation into the JSE's important illiquidity episodes.

Figure 5.2 shows the daily cumulative growth in the FTSE/JSE All Share Index over the period April 1994 to September 2009. As can be seen the market has undergone a general upward trend spurred on by growing company earnings and positive economic growth which has been interrupted by major downward spells driven by crises.

Table 5.2 which follows details the events surrounding each of the major market crises and gives a brief account of their triggers and the associated maximal loss. As noted, many of the crises, with the exception of **E**, were caused by a sudden loss of investor confidence and subsequent flight from risky assets. Most of the bear markets were characterized by precipitous and sudden downfalls. This is indicative of the knock-on sale effect discussed in Section 3.2: downturns are triggered by unexpected events which promote illiquidity and exaggerate the price impacts of stock sales. Such periods are fertile ground for the testing of liquidity measures.

During these periods of grave illiquidity, it is expected that individual stock

Key	Market Event	Reason for Crash	Market Loss
A	1997 Asian Financial Crises	Run on Thai Bhat from imminent bankruptcy led to a contagion effect that saw many of the Asian developed countries experiencing large currency shocks and devaluing asset prices. The crises resulted in large scale losses and loss of confidence in developed economies which had invested heavily in the Asia prior to the meltdown.	-27%
B	1998 Russian Default & LTCM collapse	Asian crises led to wholesale contraction in commodity prices and a severe disinvestment in Emerging Markets. Countries, like Russia, which were heavily commodity driven were thus severely hit. This coupled with a political crises, unstable exchange rate policy and mounting debt burdens led to the country defaulting on some of its debt. The knock-on effect of the crises was felt primarily by LTCM which had bet heavily on Russian spreads narrowing further. LTCM's collapse and the subsequent FED bailout had similar world-wide impacts as the current financial crises.	-42%
C	Dot-Com Bubble	Speculative bubble driven on primarily by the market's irrational belief that the internet's offering of new technology would lead to unending new growth. Many investors invested in technology firms on the blind belief that they could turn future profits and not on sound fundamentals.	-12%
D	September 11 Terrorist attacks	Following attacks in New York, most large exchanges closed. The US FED also provided additional liquidity to banks to ensure financial stability. Investors wary of wide-scale insurance losses fled risky assets for a short period.	-15%
E	2002 Market Downturn	An outbreak of accounting scandals and the knock-on effect of the Internet bubble led to a prolonged correction in asset valuations and a cyclical bear market on all global exchanges. Not really a market shock but more of an economy-wide downturn.	-35%
F	Credit crisis begins	Growing awareness that the credit cycle had peaked and increased market nervousness around the impressive boom in commodity prices led to increased volatility. The collapse of 2 of Bear Sterns largest hedge funds fuelled an immediate downward sell-off.	13%
G	Bear-Sterns bankruptcy	Unprecedented liquidity window offered by ECB to banks following mortgage security collapse signals start of credit crunch. Widespread illiquidity leads to many funds collapsing. The high profile collapse and bankruptcy of Bear Sterns makes it clear that the global financial system is in severe crises.	-20%
H	Lehman Brothers bankruptcy	As markets adjust to scope of the crises and a slight recovery is evident, news of Lehman's collapse and the collapse of other large-scale US insurers sends markets reeling.	-46%

Table 5.2: Major Market Crises

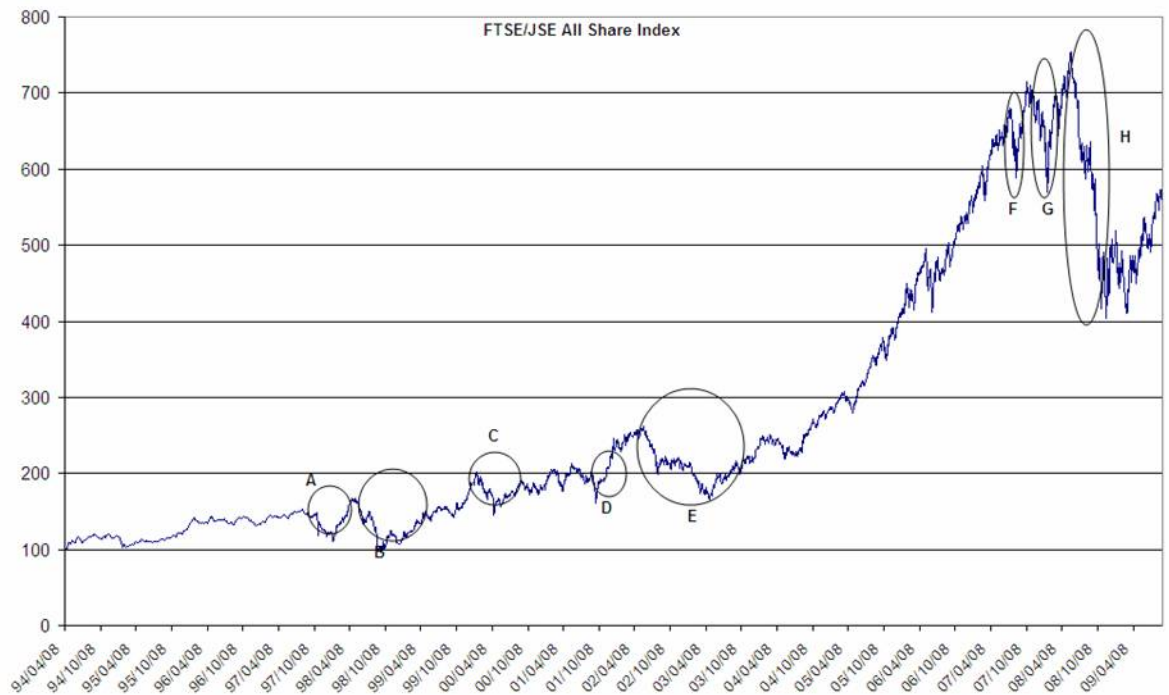


Figure 5.2: FTSE/JSE All Share Index

volumes and turnovers would initially rise as selling increases but would then drop sharply as investors do not return to risky assets. Bid-Ask spreads should widen as uncertainty and asymmetric information problems gather pace, encouraging investors to demand more to meet the other side of a trade. Volume-related return reversals should also spike upwards as in a shallow market smaller trades encounter larger price impacts. This implies that both the Amihud ratio and the P&S measure should show upward and downward trends respectively.

The graphs which follow represent the monthly level series of the liquidity measures considered across all the 4 stocks chosen. As expected volume, value and turnover show relatively similar trends over time, moving almost in sync over the period charted. The same is evident for the Amihud ratio and P&S Measure which show almost exactly opposing troughs and peaks. Although changes in a measure for a single counter can be explained by both stock specific corporate actions and economy-wide events, it seems safe to infer that if a single measure shows the same behaviour across all stocks at the same time as a known crises then this is driven by non-specific liquidity factors.

The graphs for the most liquid share, AGL, show a strikingly coordinated response to periods of illiquidity. As can be seen in Figures 5.3 to 5.5, episodes of illiquidity are borne out nicely in the local minima and maxima of the trading activity measures. Generally while liquidity in the share has increased over time, widespread liquidity crises have affected the general trading activity of the shares. While the share has other liquidity turning points which cannot be explained by market-wide liquidity events, the coordinated response of the variables makes it clear that they do shed light on liquidity, even if this information is only made available after the fact.

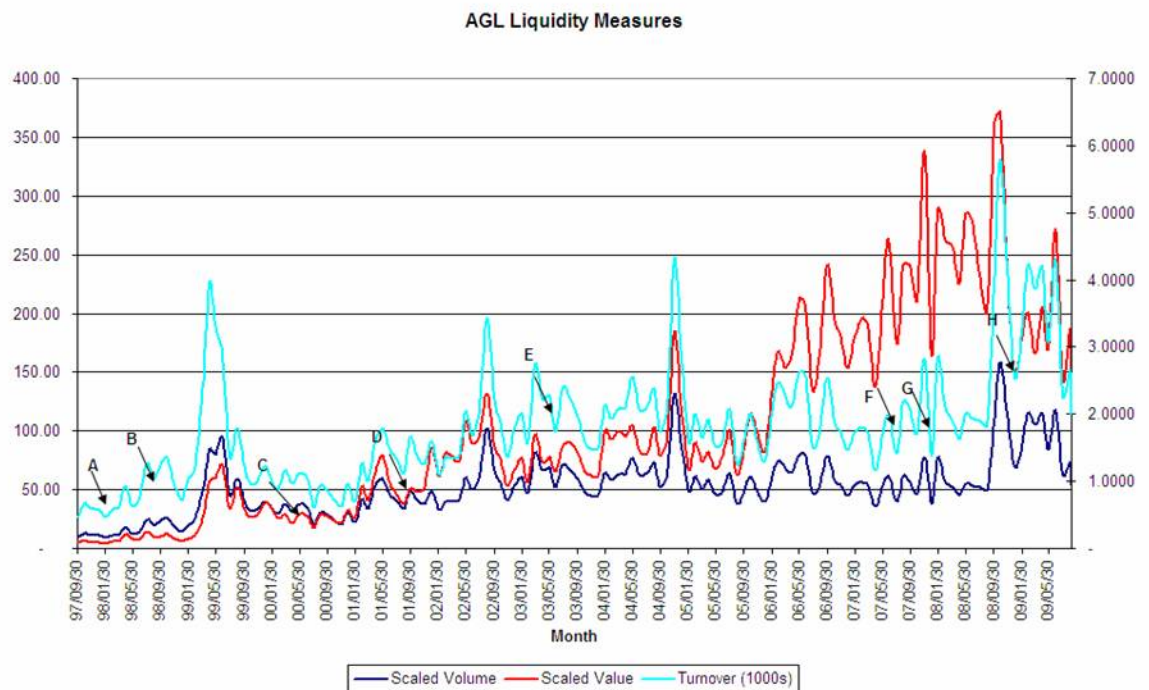


Figure 5.3: AGL Liquidity Measures - Trading Activity Variables

Overall spikes in the trading activity measures are not accompanied by a severe widening of the spread, as shown in Figure 5.4. In general the relationship seems to be lagged, which is somewhat expected – the spread is a pre-trade measure while volume is post-trade. If prices drive *ex post* volumes then it seems reasonable to conclude that spreads should widen in response to a liquidity event prior to falling volumes.

Finally, as shown in Figure 5.5, volume-related return reversals tend to be sudden and tend to reverse fairly quickly. This is in keeping with the earlier discussion regarding the speed with which a liquidity crises can take root. The return-reversals are also far more volatile than any of the other



series. This is further evidence that price impacts tend to be severely changeable. Fortunately return-reversals tend to increase dramatically almost at the same time as a crises indicating that they could be useful predictive measures. Since they are also stationary (as discussed later) they could be subject to interesting predictive time series analysis. Generally trends in the one variable are more noticeable than trends in the other, implying that it is useful to monitor both of the measures in order to keep track of a stock's market liquidity.

As is made more evident by Figure 5.6, the relationship between stock turnover and the other measures like volume can be close but is subject to some severe and interesting interruptions. This is evident for all the two large and liquid stocks AGL and PIK. Possibly – unless the liquidity conditions are severe – since these are more liquid and have more shares in issue, a large change in value or volume traded need not influence the stock turnover too much.

Once more as for AGL, the relationship between the B/A-spread and volume seems slightly lagged and negative. Interestingly the Amihud ratio for PIK, although seemingly highly flat as shown in Figure 5.8, is subject to a great

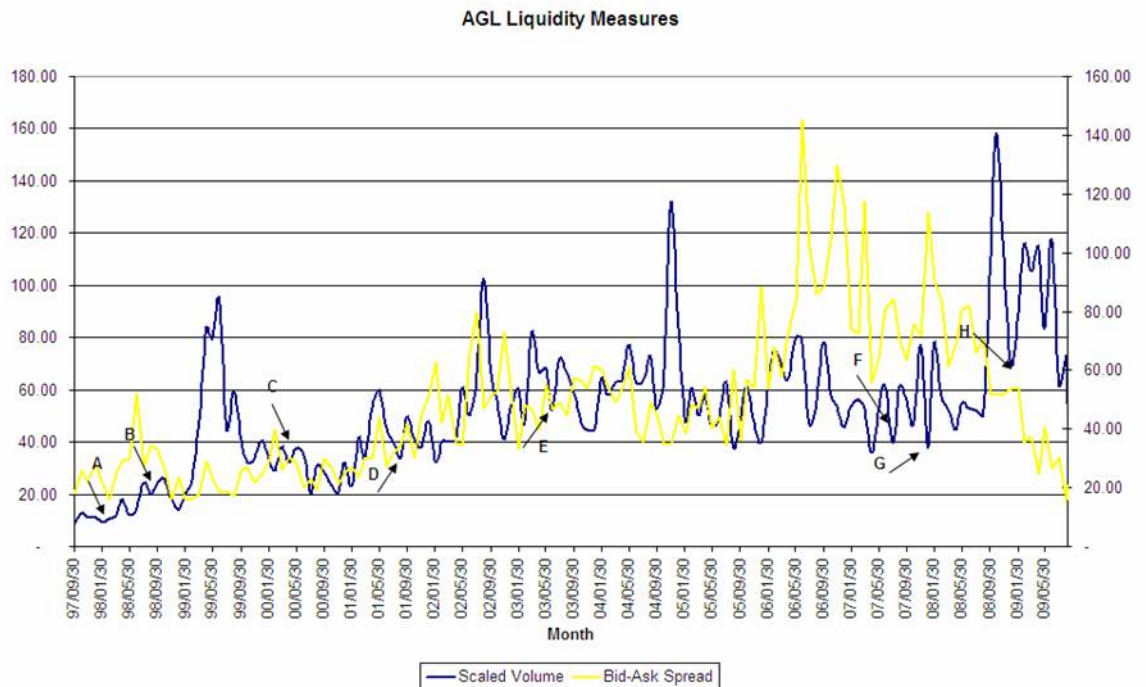


Figure 5.4: AGL Liquidity Measures - B/A Spread & Scaled Volume

deal of volatility. Fluctuations in this measure are well coordinated with that of the P&S Measure and with known liquidity crises, although there are turning points which seem largely unrelated to economy-wide events.

The graphs which follow show the measures for the 2 less liquid shares: CML and SKY.

As can be seen, for both of these shares the B/A-spread seems to play a more pivotal role in the overall liquidity of the stock than for PIK and AGL. Moreover the negative relationship between volume and the spread seems stronger and more marked.

Unfortunately the measures for these shares seem highly variable and are subject to large up and down swings from their mean. This is largely expected – as these shares trade so infrequently, any trade should have larger impacts than expected. This implies that for illiquid shares, none of the measures can be trusted as a source of pending illiquidity. Illiquidity events affect these stocks too often to gauge whether or not there are drastic shifts in their state. As these are also fairly new stocks, a longer history may shed more light on the behaviour of the measures in this case.

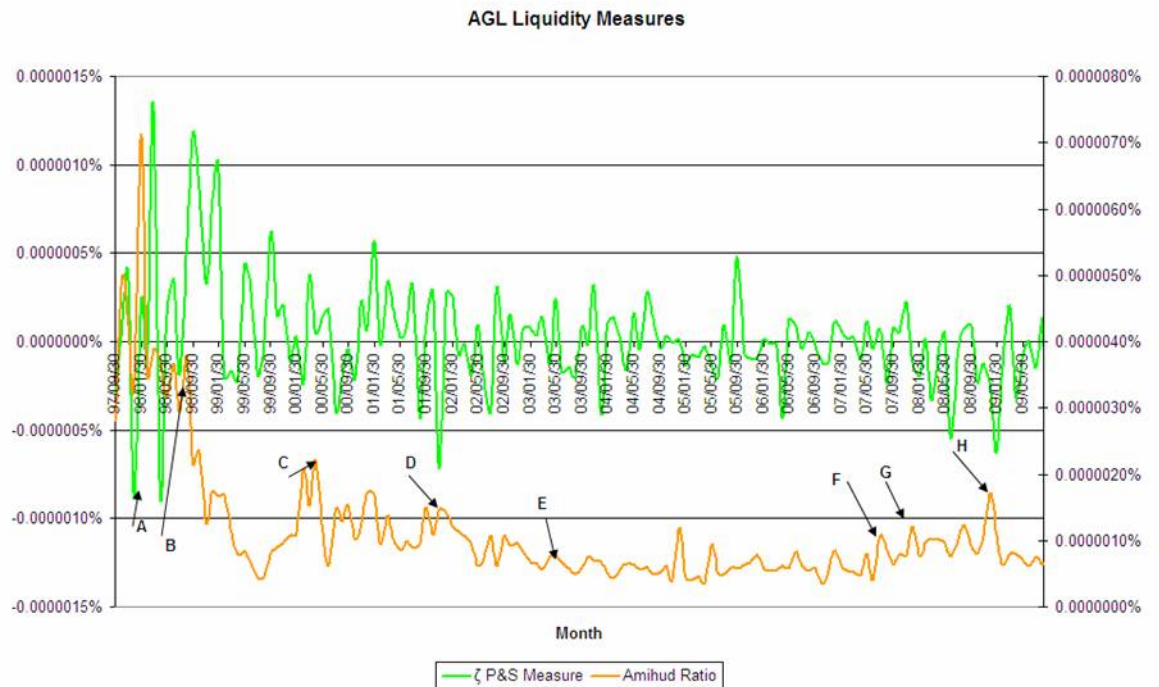


Figure 5.5: AGL Liquidity Measures - Price Impact Variables

Interestingly volume-related return reversals seem to be less variable than for the liquid shares. This is somewhat unexpected as illiquid stocks should have larger potential price reversals. However since these shares do not trade that often, the Amihud and P&S Measure may continue being stable for a long time until a new trade comes through. The fact that these measures do not indicate *potential* trade impacts but rather realized ones can be seen as a drawback for these measures.

Tables 5.3 to 5.6 show the descriptive statistics for the measures considered. Generally it is expected that, other things equal, the B/A-spread for the most liquid share be lower than that for the least liquid share. Moreover trading activity measures should decrease with greater illiquidity and vice versa with the return-reversal measures.

The results are different however. Over the data considered the mean spread for the more liquid shares, AGL and PIK, is significantly higher than that of the illiquid shares. It is also more variable. Although volume seems to decline with illiquidity, this relationship does not seem to be guaranteed as the volume for SKY is higher than that of CML. The same can be said of share turnover. Indeed value traded and the return-reversal measures

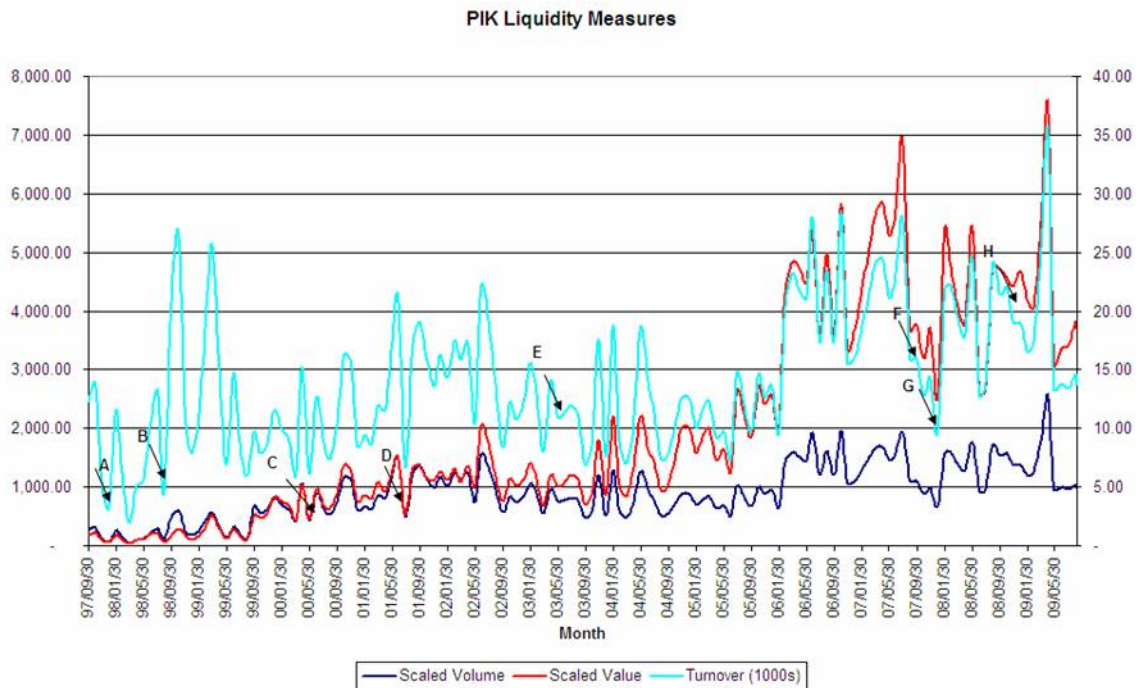


Figure 5.6: PIK Liquidity Measures - Trading Activity Variables

seem to be the only variables which increase and decrease as expected by theory. The variability of these measures also seems to be higher for the illiquid shares, reflecting the greater impact of any single trade on the stock's liquidity when the market is shallow.

The anomalous behaviour of the spread and volume cannot easily be explained. It could be caused by stock-specific events which are unrelated to liquidity. AGL for example could simply have had more sellers than buyers over the period considered, implying that liquidity for the stock may not be symmetric. Interestingly the above results remain true even when the descriptive statistics are compared across stocks over exactly the same time period. This implies that the behaviour is not sensitive to changes in regime and is endemic to the unique features of the market for each stock considered.

Although the reasons for the deviations from what is expected may be diverse, the effect does highlight the severe drawbacks in relying solely on one measure for the risk management of liquidity. At times expectations may not be met. A stock may, for example, have a very tight spread and yet incur deep price impacts which could cost a trader dearly. The fact

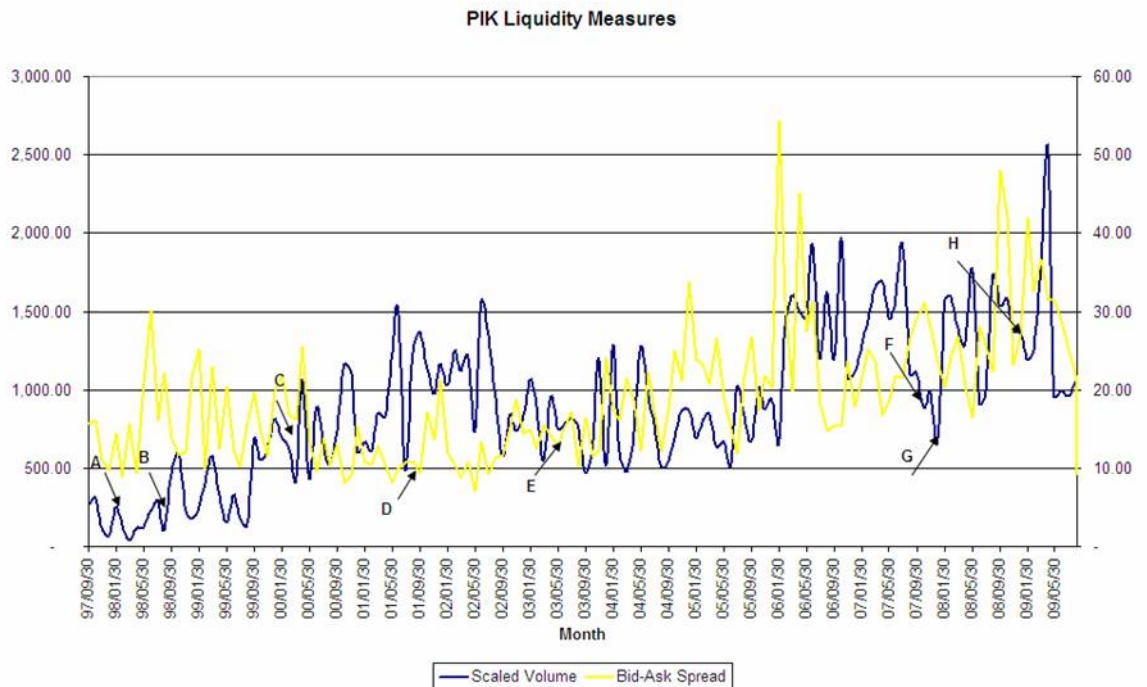


Figure 5.7: PIK Liquidity Measures - B/A Spread & Scaled Volume

that the important return-reversal measures, which bear on the most significant component of liquidity risk, endogenous risk, follow expectations is noteworthy.

As discussed earlier, undertaking a correlation analysis on non-stationary data presents several problems, the principle of which is that the conclusions drawn from the analysis is not trustworthy. In order to account for this potential problem all time series were tested for stationarity using the widely-used Dickey-Fuller test. The results of the test are presented in Tables 5.7 and 5.8 together with the test's critical values. As is visible, all of the series are stationary after 1st-differencing and the bulk of them are level stationary.

Armed with the above information and adequately differenced time series, the results of the correlation analysis in Table 5.9 can be safely interpreted. Correlations which are marked are significant with p-values under 0.5%.

The results are not overwhelmingly surprising. They indicate that, generally, the trading-activity measures tend to be closely positively correlated. As supported by the long-term graphs, these measures seem to form a natural

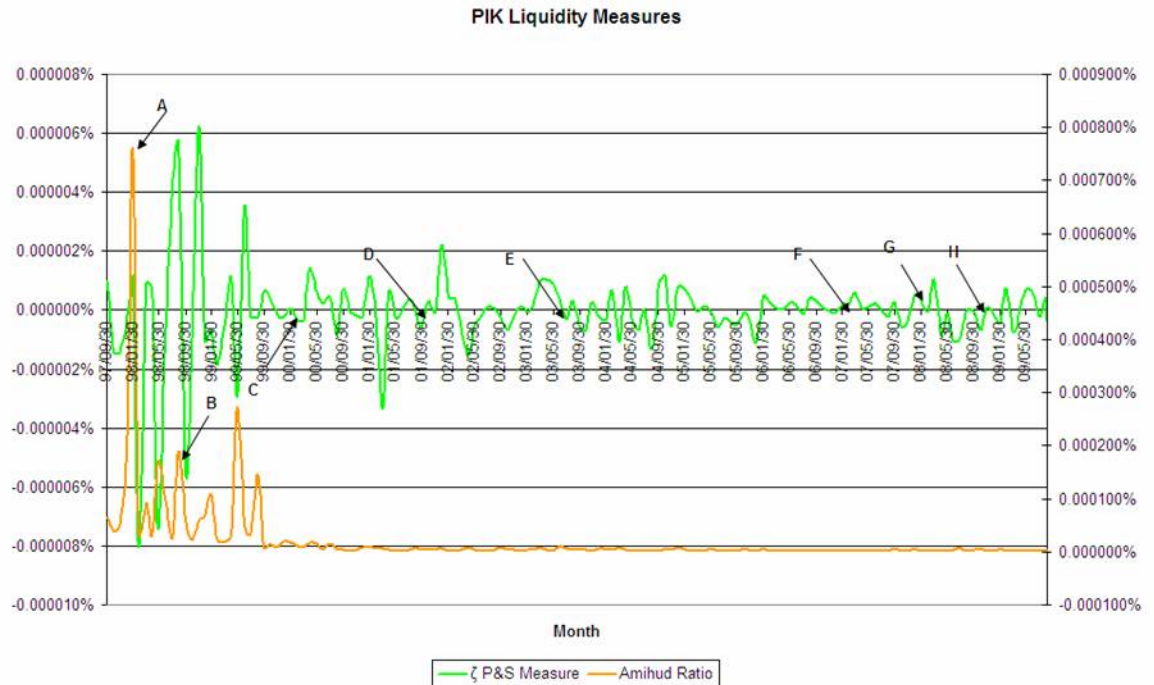


Figure 5.8: PIK Liquidity Measures - Price Impact Variables



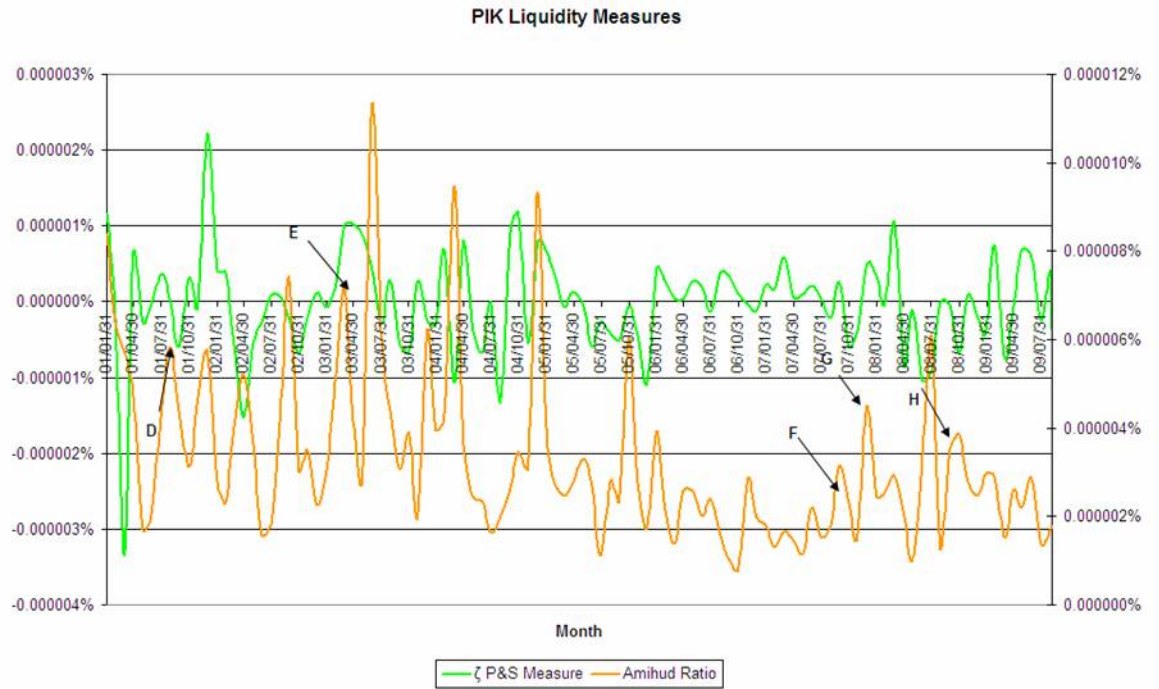


Figure 5.9: PIK Liquidity Measures - Price Impact Variables

AGL	Bid-Ask Spread	Volume	Turnover (1000s)	Value (Rm)	Amihud Ratio	ζ P & S Measure
Mean	48.10	2,593,514	1.83	51,267,569	0.00000114%	0.00000005%
Standard Error	2.08	104,892	0.08	3,413,892	0.00000008%	0.00000004%
Median	44.67	2,546,783	1.72	39,853,717	0.00000082%	0.00000001%
Mode	22.25	-	-	-	-	-
Standard Deviation	25.07	1,263,072	0.91	41,108,700	0.00000098%	0.00000047%
Sample Variance	628.64	1.60E+12	0.82	1.69E+15	0.00000000%	0.00000000%
Kurtosis	1.70	2	2.77	0	11.83	43.66
Skewness	1.20	1	1.35	1	3.07	5.13
Range	129.63	7,353,622	5.34	179,819,375	0.00000678%	0.00000510%
Minimum	15.75	430,424	0.46	2,033,471	0.00000035%	-0.00000088%
Maximum	145.38	7,784,045	5.80	181,852,846	0.00000713%	0.00000421%
Sum	6,974.17	376,059,517	265.34	7,433,797,519	0.00016505%	0.00000790%
Count	145.00	145	145.00	145	145.00	145.00
Confidence Level(95.0%)	4.12	207,328	0.15	6,747,813	0.00000016%	0.00000008%

Table 5.3: AGL - Descriptive Statistics

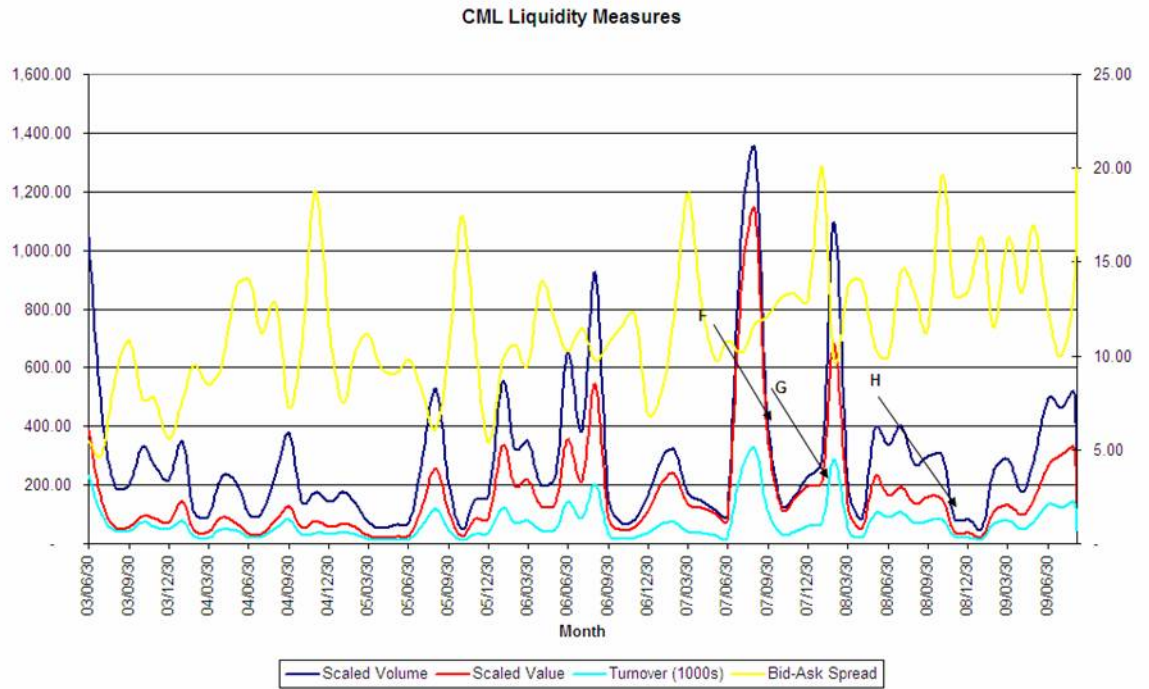


Figure 5.10: CML Liquidity Measures - Trading Activity Variables

PIK	Bid-Ask Spread	Volume	Turnover (1000s)	Value (Rm)	Amihud Ratio	$\zeta$ P & S Measure
Mean	18.97	639,183	14.19	1,440,822	0.00002172%	-0.00000006%
Standard Error	0.70	27,738	0.50	103,611	0.00000608%	0.00000013%
Median	17.17	613,188	13.40	930,059	0.00000346%	-0.00000005%
Mode	10.50	-	-	-	-	-
Standard Deviation	8.40	334,014	5.97	1,247,644	0.00007324%	0.00000151%
Sample Variance	70.63	1.12E+11	35.70	1.56E+12	0.00000000%	0.00000000%
Kurtosis	2.64	0	0.32	-0	74.06	12.84
Skewness	1.35	0	0.65	1	7.84	-1.15
Range	47.24	1,764,614	33.56	5,296,154	0.00076104%	0.00001429%
Minimum	7.09	29,668	1.90	26,308	0.00000077%	-0.00000803%
Maximum	54.33	1,794,282	35.45	5,322,463	0.00076181%	0.00000626%
Sum	2,751.33	92,681,472	2,058.04	208,919,204	0.00314993%	-0.00000898%
Count	145.00	145	145.00	145	145.00	145.00
Confidence Level(95.0%)	1.38	54,827	0.98	204,795	0.00001202%	0.00000025%

Table 5.4: PIK - Descriptive Statistics

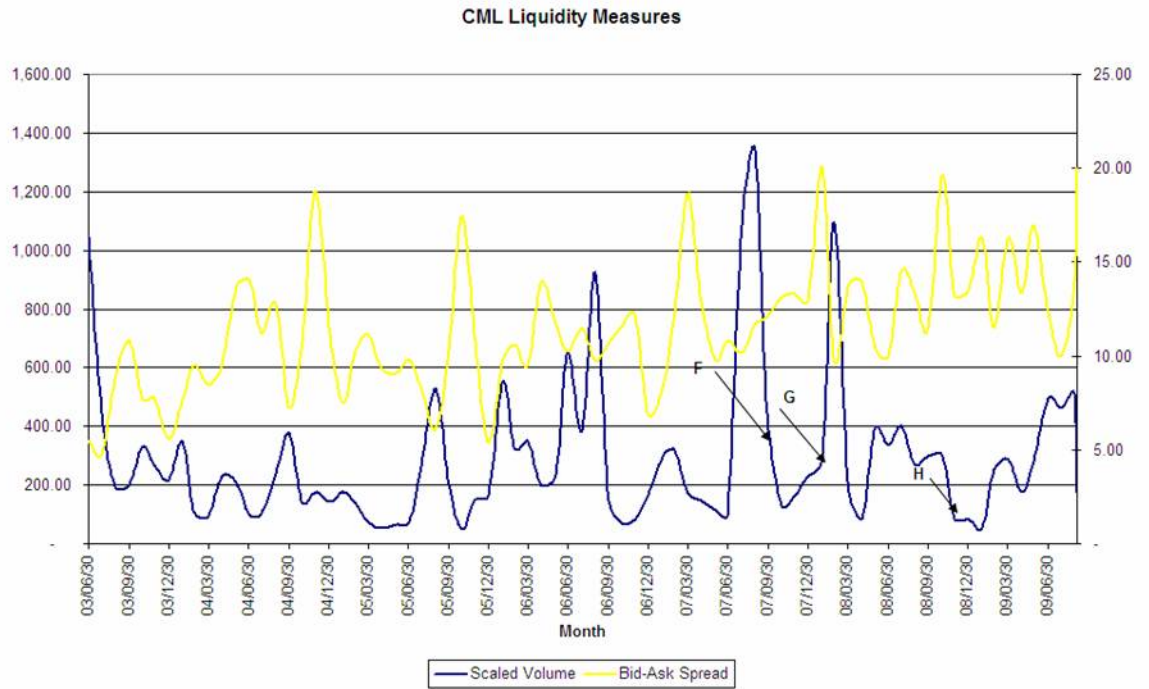


Figure 5.11: CML Liquidity Measures - B/A Spread & Scaled Volume

CML	Bid-Ask Spread	Volume	Turnover (1000s)	Value (Rm)	Amihud Ratio	$\zeta$ P & S Measure
Mean	11.44	382,095	1.08	217,859	0.00005663%	-0.00000052%
Standard Error	0.39	38,837	0.11	27,942	0.00001179%	0.00000022%
Median	10.96	278,696	0.74	148,629	0.00002224%	-0.00000003%
Mode	10.81	-	-	-	-	-
Standard Deviation	3.41	338,571	0.96	243,597	0.00010282%	0.00000189%
Sample Variance	11.64	1.15E+11	0.92	5.93E+10	0.00000000%	0.00000000%
Kurtosis	0.43	6	5.75	13	10.31	9.68
Skewness	0.58	2	2.27	3	3.24	-2.31
Range	15.31	1,706,285	4.85	1,467,616	0.00053431%	0.00001385%
Minimum	4.74	65,320	0.17	26,025	0.00000261%	-0.00001030%
Maximum	20.05	1,771,605	5.02	1,493,640	0.00053692%	0.00000354%
Sum	869.78	29,039,191	82.06	16,557,251	0.00430394%	-0.00003948%
Count	76.00	76	76.00	76	76.00	76.00
Confidence Level(95.0%)	0.78	77,367	0.22	55,664	0.00002349%	0.00000043%

Table 5.5: CML - Descriptive Statistics



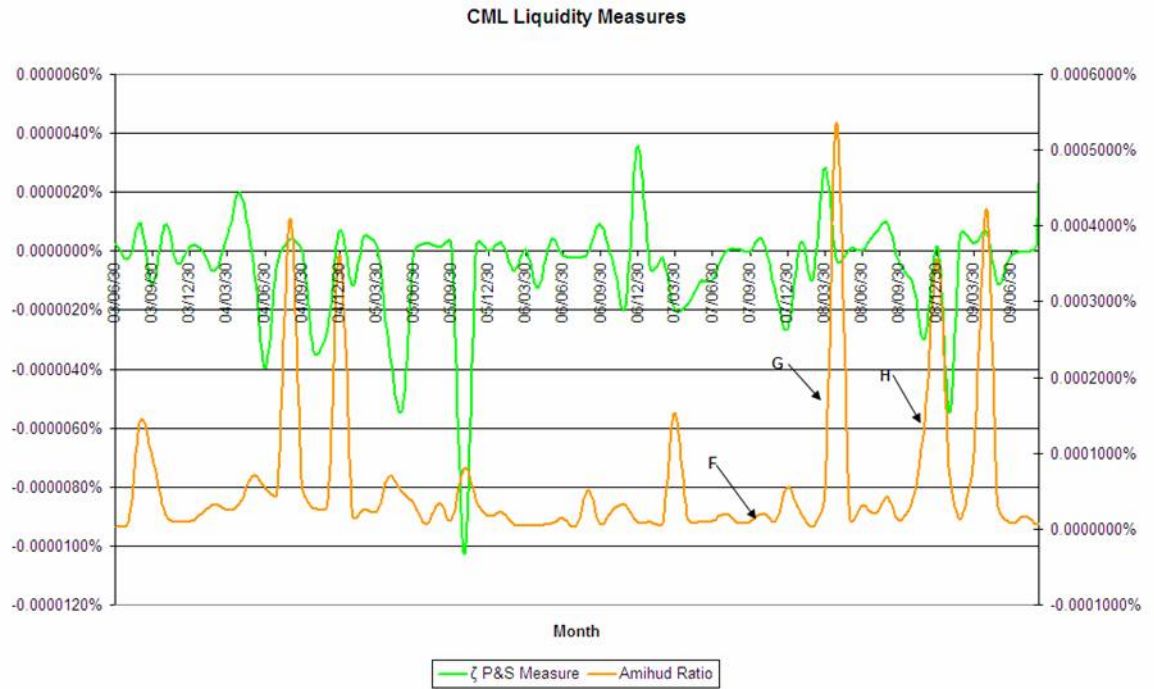


Figure 5.12: CML Liquidity Measures - Price Impact Variables

SKY	Bid-Ask Spread	Volume	Turnover (1000s)	Value (Rm)	Amihud Ratio	$\zeta$ P & S Measure
Mean	6.90	868,210	1.80	144,933	0.00010287%	-0.00000027%
Standard Error	0.75	178,442	0.37	35,261	0.00003534%	0.00000022%
Median	6.25	598,848	1.24	80,779	0.00002229%	-0.00000008%
Mode	-	-	-	-	-	-
Standard Deviation	3.82	909,881	1.90	179,797	0.00018019%	0.00000110%
Sample Variance	14.59	8.28E+11	3.63	3.23E+10	0.00000000%	0.00000000%
Kurtosis	5.57	12	11.80	11	7.57	2.20
Skewness	2.28	3	3.02	3	2.72	-1.21
Range	16.30	4,553,613	9.54	878,776	0.00076730%	0.00000507%
Minimum	2.35	73,614	0.15	4,553	0.00000182%	-0.00000321%
Maximum	18.65	4,627,227	9.69	883,329	0.00076912%	0.00000187%
Sum	179.46	22,573,465	46.89	3,768,259	0.00267454%	-0.00000689%
Count	26.00	26	26.00	26	26.00	26.00
Confidence Level(95.0%)	1.54	367,509	0.77	72,621	0.00007278%	0.00000044%

Table 5.6: SKY - Descriptive Statistics

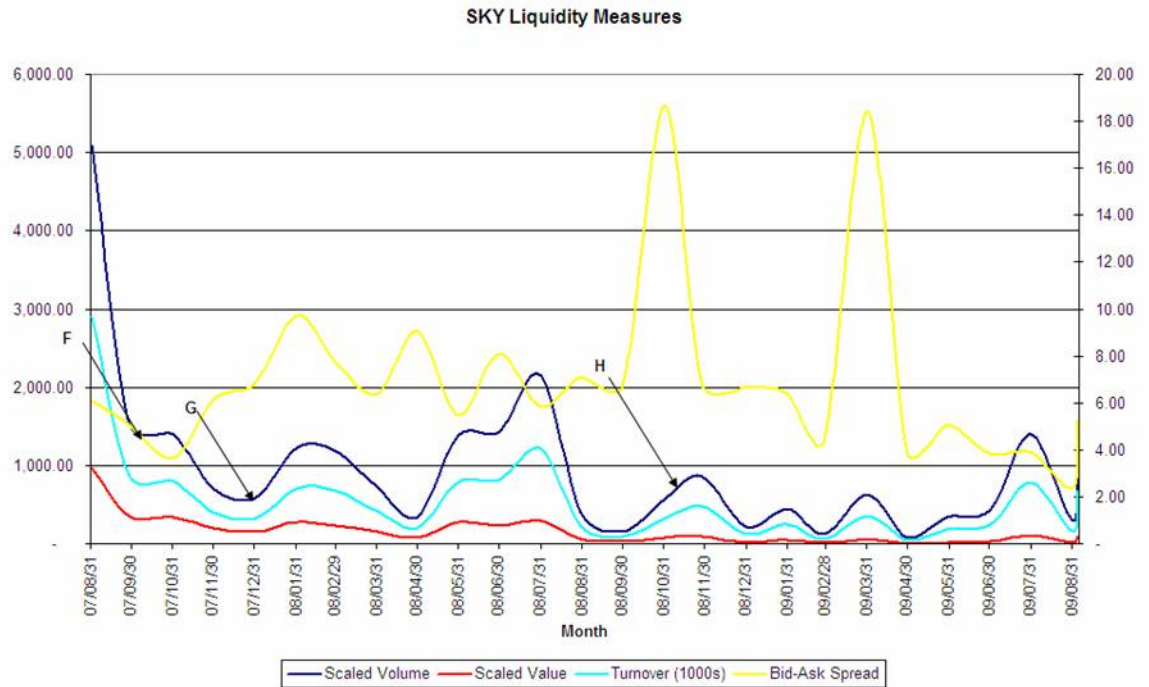


Figure 5.13: SKY Liquidity Measures - Trading Activity Variables

Testing Lags	4 lags	4 lags	2 lags	1 lag
ADF Test Statistic	AGL	PIK	CML	SKY
P&S Measure	-2.836679	-4.547355	-5.880174	-4.68718
Amihud Ratio	-6.649882	-9.535022	-5.195272	-2.194317
Scaled Volume	-3.982878	-3.230956	-5.552362	-3.468753
Value	-2.333942	-3.058404	-4.569437	-3.273936
Bid/Ask Spread	-0.8812	-3.59896	-5.244469	-3.030987
Scaled Value	-2.333942	-3.058404	-4.569437	-3.273936
Turnover	-3.920962	-3.595455	-5.562699	-3.456711
Volume	-3.982878	-3.230956	-5.552362	-3.468753
1% Critical Value	-4.0268			
5% Critical Value	-3.4428			
10% Critical Value	-3.1458			

Table 5.7: Dickey-Fuller Unit Root Test Results

ADF Test Statistic	AGL	PIK	CML	SKY
ADF Test Statistic	AGL	PIK	CML	SKY
P&S Measure	1st Difference	Level	Level	Level
Amihud Ratio	Level	Level	Level	1st Difference
Scaled Volume	Level	Level	Level	Level
Value	1st Difference	1st Difference	Level	Level
Bid/Ask Spread	1st Difference	Level	Level	1st Difference
Scaled Value	1st Difference	1st Difference	Level	Level
Turnover	Level	Level	Level	Level
Volume	Level	Level	Level	Level

Table 5.8: Dickey-Fuller Unit Root Test Results - Level vs Differenced Series

AGL	dP&S	dValue	dBidAsk	dScalVal	Volume	Turnover	Amihud	ScalVol
dP&SMeasure	1	-0.01	-0.03	-0.01	0.03	0.04	0.07	0.03
dValue	-0.01	1	<b>-0.2</b>	<b>1</b>	<b>0.26</b>	<b>0.26</b>	-0.07	<b>0.26</b>
dBidAsk	-0.03	<b>-0.2</b>	1	<b>-0.2</b>	-0.07	-0.08	0.05	-0.07
dScalVal	-0.01	<b>1</b>	<b>-0.2</b>	1	<b>0.26</b>	<b>0.26</b>	-0.07	<b>0.26</b>
Volume	0.03	<b>0.26</b>	-0.07	<b>0.26</b>	1	<b>0.98</b>	<b>-0.55</b>	<b>1</b>
Turnover	0.04	<b>0.26</b>	-0.08	<b>0.26</b>	<b>0.98</b>	1	<b>-0.45</b>	<b>0.98</b>
Amihud Ratio	0.07	-0.07	0.05	-0.07	<b>-0.55</b>	<b>-0.45</b>	1	<b>-0.55</b>
Scaled Volume	0.03	<b>0.26</b>	-0.07	<b>0.26</b>	<b>1</b>	<b>0.98</b>	<b>-0.55</b>	1
PIK	dValue	dScalVal	Bid-Ask	Volume	Turnover	Amihud	P&S	ScalVol
dValue	<b>1</b>	1	-0.09	<b>0.35</b>	<b>0.4</b>	-0.01	0	<b>0.35</b>
dScalVal	<b>1</b>	1	-0.09	<b>0.35</b>	<b>0.4</b>	-0.01	0	<b>0.35</b>
Bid-Ask	-0.09	-0.09	1	<b>0.3</b>	<b>0.26</b>	-0.07	0.05	<b>0.3</b>
Volume	<b>0.35</b>	<b>0.35</b>	<b>0.3</b>	1	<b>0.87</b>	<b>-0.37</b>	0.07	<b>1</b>
Turnover	<b>0.4</b>	<b>0.4</b>	<b>0.26</b>	<b>0.87</b>	1	<b>-0.23</b>	0.01	<b>0.87</b>
Amihud Ratio	-0.01	-0.01	-0.07	<b>-0.37</b>	<b>-0.23</b>	1	-0.02	<b>-0.37</b>
P&S Measure	0	0	0.05	0.07	0.01	-0.02	1	0.07
Scaled Volume	<b>0.35</b>	<b>0.35</b>	<b>0.3</b>	<b>1</b>	<b>0.87</b>	<b>-0.37</b>	0.07	1
CML	Bid-Ask	Volume	Turnover	Value	Amihud	P&S	ScalVol	ScalVal
Bid-Ask	1	-0.21	-0.16	-0.08	0.21	<b>-0.26</b>	-0.21	-0.08
Volume	-0.21	1	<b>0.99</b>	<b>0.94</b>	-0.22	0.21	<b>1</b>	<b>0.94</b>
Turnover	-0.16	<b>0.99</b>	1	<b>0.94</b>	-0.22	0.21	<b>0.99</b>	<b>0.94</b>
Value	-0.08	<b>0.94</b>	<b>0.94</b>	1	-0.22	0.16	<b>0.94</b>	<b>1</b>
Amihud Ratio	0.21	-0.22	-0.22	-0.22	1	0	-0.22	-0.22
P&S Measure	<b>-0.26</b>	0.21	0.21	0.16	0	1	0.21	0.16
Scaled Volume	-0.21	<b>1</b>	<b>0.99</b>	<b>0.94</b>	-0.22	0.21	1	<b>0.94</b>
Scaled Value	-0.08	<b>0.94</b>	<b>0.94</b>	<b>1</b>	-0.22	0.16	<b>0.94</b>	1
SKY	dBidAsk	dAmihud	Volume	Turnover	Value	P&S	ScalVol	ScalVal
dBidAsk	1	<b>0.4</b>	0.02	0.02	-0.01	0.29	0.02	-0.01
dAmihud	<b>0.4</b>	1	0.24	0.23	0.09	0.11	0.24	0.09
Volume	0.02	0.24	1	<b>1</b>	<b>0.85</b>	0.37	<b>1</b>	<b>0.85</b>
Turnover	0.02	0.23	<b>1</b>	<b>1</b>	<b>0.86</b>	0.37	<b>1</b>	<b>0.86</b>
Value	-0.01	0.09	<b>0.85</b>	<b>0.86</b>	1	0.3	<b>0.85</b>	<b>1</b>
P&S Measure	0.29	0.11	0.37	0.37	0.3	1	0.37	0.3
Scaled Volume	0.02	0.24	<b>1</b>	<b>1</b>	<b>0.85</b>	0.37	1	<b>0.85</b>
Scaled Value	-0.01	0.09	<b>0.85</b>	<b>0.86</b>	<b>1</b>	0.3	<b>0.85</b>	1

Table 5.9: Correlation Analysis Results

grouping and carry similar information on liquidity. The relationship is stronger for the illiquid shares.

Overall the Bid-Ask spread has a very low correlation with the other measures. Its correlations are also not stable across the stocks, changing from positive in some cases to negative. This is tentative support for the idea that it measures a different aspect of liquidity, one related specifically to exogenous risk.

The price-impact measures have very low correlations across the board, with the P&S measure bearing the lowest cross-correlation to any of the variables. The Amihud ratio, however, tends to be negatively related to measures of trading activity. This makes intuitive sense – as markets deepen, price impacts linked to return-reversals should fall.

While in general the correlations between the same variables is not stable across different stocks, it seems safe to conclude that liquidity measures do indeed fall into separate groupings, with each group pertaining to a particular aspect of liquidity. While the measures are not substitutable, at the same time they do not seem to be exhaustive in that they do not account

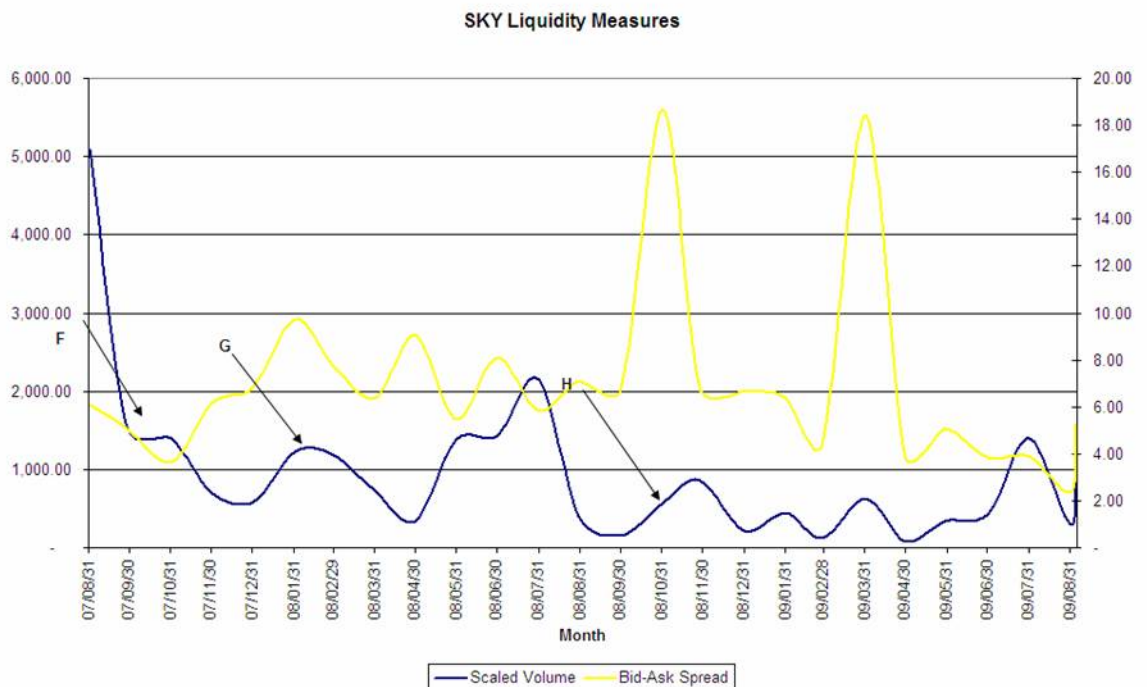


Figure 5.14: SKY Liquidity Measures - B/A Spread & Scaled Volume

for the totality of liquidity risk.

The results of the Factor Analysis give some credence to the above conclusion. In all cases, the four factors extracted by way of a principle component methodology, account for over 96% of the total variability in the original data set. Combined, the factors account for on average 97% of the variation in the individual variables as well <sup>3</sup>. The factors are thus variables which very closely represent the information of the original variables.

As shown in Table 5.10, a similar pattern is exhibited across all the stocks: Factor 1, which accounts for the largest variability, has the highest loading on trading activity measures; Factor 2, with one exception, has the highest loading on the P & S Measure, while Factor 3 is most correlated to the price-impact measures and Factor 4 is generally positively correlated to the spread.

If the total variability in the set of original variables is thought to capture the aggregate variability in a stock's liquidity then this implies that trading activity accounts for the bulk of aggregate change in a share's liquidity,

<sup>3</sup>This is indicated by each variable's communality

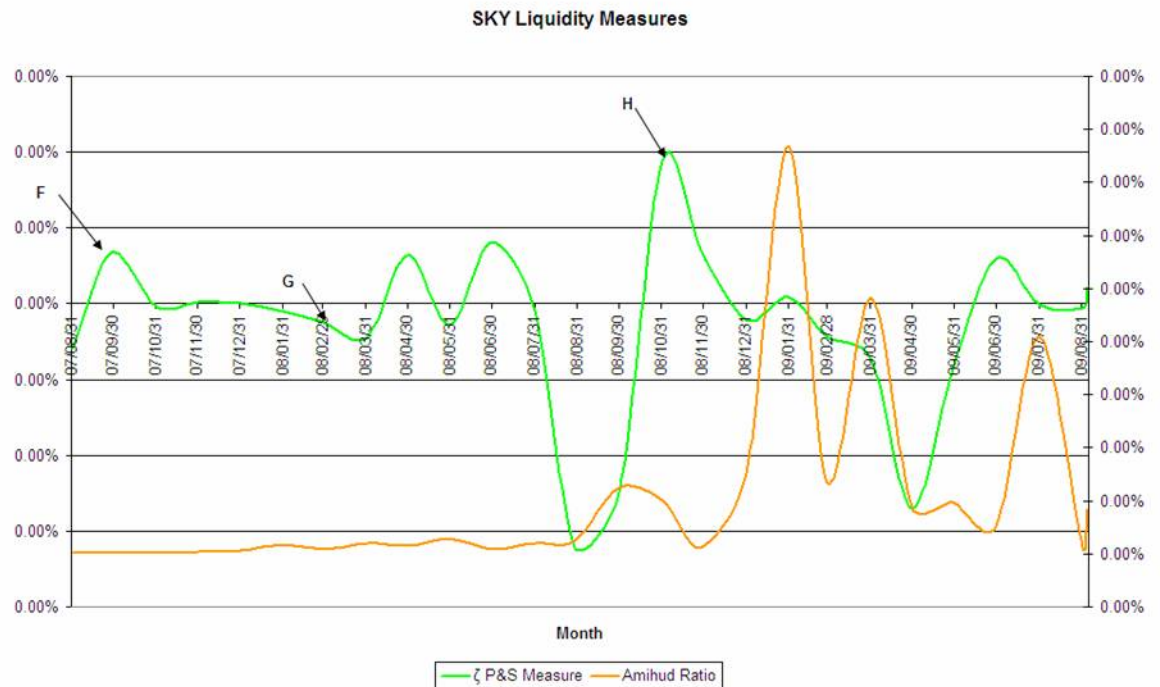


Figure 5.15: SKY Liquidity Measures - Price Impact Variables

AGL				
Factor Loadings	Factor 1	Factor 2	Factor 3	Factor 4
Bid-Ask Spread	0.054458	<b>0.953602</b>	0.108234	0.164052
Turnover	<b>0.970146</b>	0.074962	0.037495	0.174484
Amihud Ratio	-0.299446	-0.173785	-0.030816	<b>-0.935547</b>
P&S Measure	-0.049615	-0.087293	<b>-0.994242</b>	-0.027073
Scaled Volume	<b>0.938227</b>	0.103809	0.054264	0.284962
Scaled Value	0.667937	0.655899	0.022727	0.102592
<b>Expl.Var</b>	2.362687	1.393777	1.006048	1.025068
Prp.Totl	0.393781	0.232296	0.167675	0.170845
AGL	Eigenvalue	% Total Variance	Cumulative Eigenvalue	Cumulative %
Factor 1	3.165354	52.755899	3.165354	52.755899
Factor 2	1.108342	18.47237	4.273696	71.228269
Factor 3	0.878057	14.63429	5.151754	85.862559
Factor 4	0.635826	10.597105	5.78758	96.459664
PIK				
Factor Loadings	Factor 1	Factor 2	Factor 3	Factor 4
Bid-Ask Spread	0.189255	0.023437	0.015593	<b>0.975006</b>
Turnover	<b>0.948134</b>	-0.010617	0.050906	0.051236
Amihud Ratio	-0.171933	-0.007804	<b>-0.983844</b>	-0.019929
P&S Measure	0.019736	<b>0.999458</b>	0.007754	0.022095
Scaled Volume	<b>0.939164</b>	0.039552	0.222328	0.122804
Scaled Value	<b>0.851678</b>	0.024355	0.13325	0.392408
<b>Expl.Var</b>	2.572111	1.001797	1.03803	1.123212
Prp.Totl	0.428685	0.166966	0.173005	0.187202
PIK	Eigenvalue	% Total Variance	Cumulative Eigenvalue	Cumulative %
Factor 1	3.003342	50.055698	3.003342	50.055698
Factor 2	1.005569	16.759484	4.008911	66.815182
Factor 3	0.952024	15.867067	4.960935	82.682249
Factor 4	0.774215	12.903591	5.73515	95.58584
CML				
Factor Loadings	Factor 1	Factor 2	Factor 3	Factor 4
Bid-Ask Spread	-0.076938	-0.129524	0.101368	<b>-0.983065</b>
Turnover	<b>0.982594</b>	0.093305	-0.08922	0.067531
Amihud Ratio	-0.130648	0.012939	<b>0.986261</b>	-0.100205
P&S Measure	0.115034	0.984918	0.013061	0.128508
Scaled Volume	<b>0.977484</b>	0.088192	-0.086898	0.120752
Scaled Value	<b>0.970699</b>	0.0555	-0.092946	-0.019169
<b>Expl.Var</b>	2.899441	1.006571	1.007307	1.01248
Prp.Totl	0.48324	0.167762	0.167884	0.168747
CML	Eigenvalue	% Total Variance	Cumulative Eigenvalue	Cumulative %
Factor 1	3.097685	51.628076	3.097685	51.628076
Factor 2	1.182857	19.714283	4.280542	71.342359
Factor 3	0.991174	16.519568	5.271716	87.861927
Factor 4	0.654084	10.901398	5.925799	98.763325
SKY				
Factor Loadings	Factor 1	Factor 2	Factor 3	Factor 4
Bid-Ask Spread	-0.040239	0.119698	-0.113685	<b>0.985449</b>
Turnover	<b>0.992361</b>	0.066808	0.055795	-0.029636
Amihud Ratio	-0.134473	-0.01584	<b>-0.983849</b>	0.114797
P&S Measure	0.084371	<b>0.989229</b>	0.015683	0.118377
Scaled Volume	<b>0.991732</b>	0.06828	0.052043	-0.029772
Scaled Value	<b>0.967364</b>	0.031523	0.174384	-0.023287
<b>Expl.Var</b>	2.930928	1.003271	1.01736	1.000607
Prp.Totl	0.488488	0.167212	0.16956	0.166768
SKY	Eigenvalue	% Total Variance	Cumulative Eigenvalue	Cumulative %
Factor 1	3.056331	50.938852	3.056331	50.938852
Factor 2	1.301299	21.688313	4.35763	72.627165
Factor 3	0.928925	15.482086	5.286555	88.10925
Factor 4	0.665612	11.093525	5.952167	99.202776

Table 5.10: Factor Analysis Results

followed by the price-impacts and then the spread. This seems counter-intuitive given the little attention that the trading-activity measures have received in the literature.

Generally each Factor, for a particular stock, is only related to a specific type of variable. There are no cases where a factor is both significantly related, for example, to the spread and a trading activity measure. This is also true of AGL where factor loadings are most different to the other stocks – here Factor 4 seems to be a price-impact measure while Factor 2 accounts for spread-based risk. Thus even though the signs of the correlations between a measure and a particular Factor is not stable and changes across the different stocks, the fact that different measures capture different liquidity information is clear.

Although the robustness of the above analysis is not guaranteed and, as stated earlier, the analysis is in no way an objective test of the measures' capability, the over-arching conclusion which can be drawn is that the liquidity measures considered are sensitive to changes in market-wide liquidity and thus do capture liquidity-related information. Moreover the different measures proxy for different aspects of liquidity risk. No single descriptive measure seems to capture the totality of these aspects. This complicates the management and forecasting of liquidity risk and makes it imperative to use either an accurate, fully integrated liquidity risk measure or an array of specialised measures.

### 5.2.2 L-VaR Analysis

L-VaR presents the opportunity to holistically measure market-risk and liquidity risk in a single integrated measure. While the L-VaR models put forward in the literature differ in their scope and complexity, the theory behind all of them and their respective derivations indicate that they all share a latent ability to capture the totality of liquidity risk. The model derivations display great detail and elegance in addressing the array of aspects which affect liquidity risk.

Despite this detail, the literature has not thoroughly tested any of the models. The verdict as to whether these integrated measures are actually more accurate and complete has thus not yet been finalised. This makes the backtesting analysis presented below more significant.

The analysis which follows begins with a cursory overview of the market micro-structure data for each of the stocks and then discusses the backtesting

<b>Summary Liquidity Statistics</b>	<b>AGL</b>	<b>PIK</b>	<b>CML</b>	<b>SKY</b>
Number of Trade Days	125	125	125	125
Trade points - Half-Hour	366	366	366	366
Average Trade Price	23,759	3,491	259	15
Average Trade Volume	126,58	32,048	11,112	18,584
Average Bid	25,001	3,511	312	16
Average Bid Volume	168,1507	358,416	21,500	37,532
Average Ask	24,973	3,523	307	13
Average Ask Volume	1,432,191	429,641	29,654	14,934
Number of Zero Trade Price points	18	18	231	294
Number of Zero Bid trade points	0	15	204	286
Number of Zero Ask trade points	0	15	209	304
Total Volume Traded	46,329,630	11,729,532	4,011,364	6,764,620
Total Volume Bid	615,431,749	131,180,264	7,761,600	13,661,893
Total Volume Ask	524,182,003	157,248,455	10,704,933	5,435,978
Price change over Period	3.49%	3.27%	8.96%	-9.71%

Table 5.11: Summary Micro-Structure Statistics

graphs. This is concluded with an assessment of each model's accuracy using well-known backtesting tests and a sensitivity analysis for the key inputs of each model. The sensitivity analysis helps in determining each model's critical vulnerability and source of estimation risk.

Table 5.11 highlights the preliminary features of the market micro-structure for each of the stocks using the half-hour tick data discussed earlier. As is clear, average volumes tend to decrease as shares become more illiquid. The exception is SKY which has marginally higher trade volumes than CML over the period considered. Generally the price pressure over the 125 days is positive, except for SKY where prices fell.

As is common in an order-based market like the JSE, transaction, bid and ask prices also seem to decrease with illiquidity indicating that illiquid shares have lower demand and thus trade at "lower" absolute levels. This is echoed in the number of zero trade, bid and ask points, which represent the number of half-hour ticks at which the market quoted no firm price to buy or sell or executed no trade. The larger number of zero data points for the illiquid shares is significant as it makes their data set more discontinuous, thereby making parameter estimation more volatile and difficult. It points to the fact that measuring risk for illiquid stocks is complicated by a lack of data to make accurate and up-to-date estimates.

The graphs which follow display the various L-VaR estimates over time against the realised profits and losses borne off the delta-hedging trading strategy. Models are loosely grouped together in a single graph based on their similarity. All the VaR model forecasts are displayed as line graphs



against the realized portfolio profits and losses which are scatter plots and a black line representing its 2-point moving average (henceforth referred to as the 2-point MA).

## Standard VaR Models

Figure 5.16 displays the backtesting results of the standard, non-liquidity adjusted VaR models for AGL. What is immediately apparent from the graph is that the GARCH VaR is far more volatile than any of the other models. This trend is true across all the stocks and arises primarily because the GARCH estimate of return volatility is far more changeable than the standard volatility estimate or indeed the EWMA volatility. The GARCH VaR forecast is also far more responsive to changes in the realized profit and loss, as shown by the marked co-movement between it and the 2-point MA which rise and fall simultaneously.

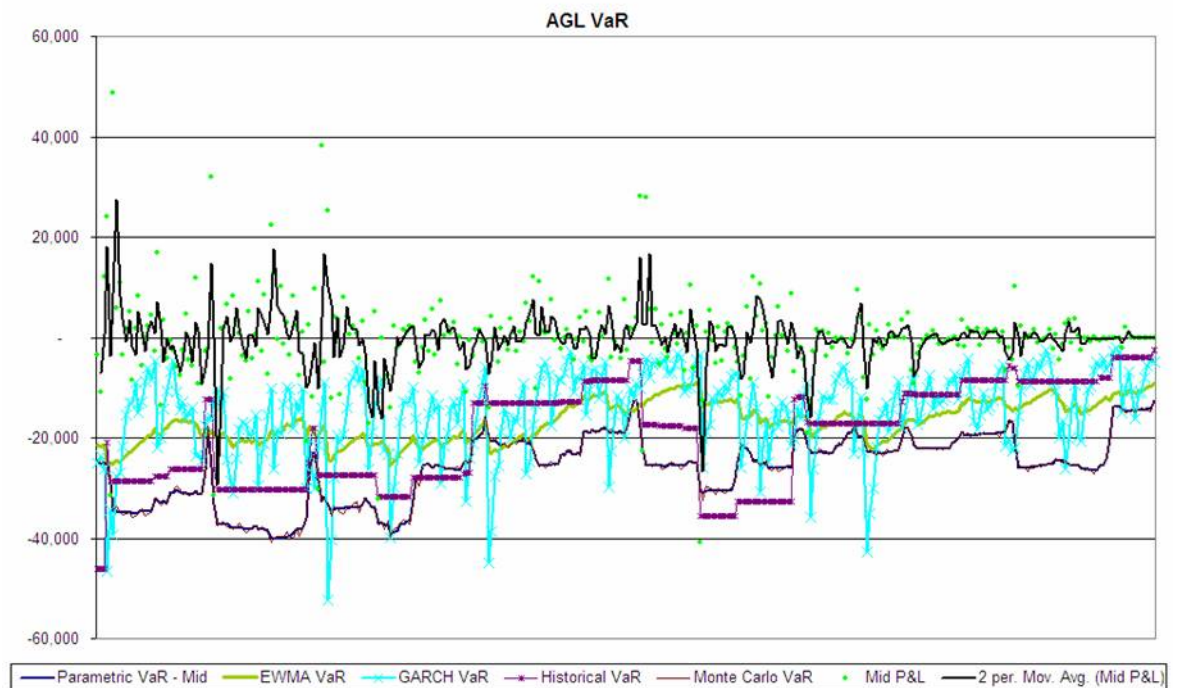


Figure 5.16: AGL Standard VaR Models

Unlike the GARCH model, the EWMA VaR series displays a much more smooth estimate of loss than any of the other models, changing gradually to match changes in the realized portfolio return. This is consistent with the choice of  $\lambda$  used in the model, a lower level would undoubtedly result in a

more volatile series. Similarly, a higher choice of  $\beta$  in the GARCH regime would have resulted in a VAR estimate which would be less responsive to changes in the portfolio return series.

As shown, the Monte Carlo VaR and standard parametric VaR almost coincide and are difficult to distinguish. This is principally because they are both based on the same normal distributional return assumptions. Both of the models offer forecasts which are more stable than either of the other models but exhibit marked step-like behaviour. This is indicative of the non-gradual up-take of new information to these models which generally only change in response to significant changes in the return input.

As with Historical VaR, estimates of the interior P & L distribution in these models tend to be clustered together and are prone to sudden changes as return inputs roll out of the base from which parameter estimates are computed. Volatility clustering is a well-known feature of high-frequency return data and would impart the observed step-like behaviour to the models over time. The fact that most of the models, show such remarkable trend similarity to the Historical VaR remains interesting however.

Overall the models displayed in Figure 5.16 display relatively good coverage of the realized profit and loss distribution. The EWMA VaR seems the least conservative and while the GARCH VaR is more sensitive to new information, it tends to overreact, making its forecasts more conservative than required.

The graph displayed in Figure 5.17, shows the same VaR models as discussed above for PIK. The most striking feature of the graph is the greater volatility and absolute levels of the Monte Carlo VaR series which is plotted against the right-hand, secondary axis. It is both larger than the same model for AGL and far larger than the other PIK VaR forecasts.

Generally, however, all of the trends displayed for AGL forecasts are preserved. Indeed the close co-movement between Historical VaR and the standard parametric VaR is even more apparent. Both clearly exhibit step-like properties. As with AGL, the EWMA VaR is subject to sharp declines from which it smoothly recovers upwards.

For both AGL and PIK, the GARCH VaR tends to provide larger estimates of loss even when the profit and loss tends to spike upwards. Thus paradoxically when the portfolio displays greater upwards volatility, the VaR estimate of loss becomes larger. This is something which the Historical VaR does not suffer from.

Figures 5.18 and 5.19 display the same models as applied to CML and SKY. Generally the trends between the VaR models are preserved but as these are illiquid stocks there are some caveats relating to the VaR models' level and volatility.

As noted in the discussion of Table 5.11, CML and SKY, being illiquid tend to have more zero trade data points than the other stocks. The standard practice in financial firms in cases where a security's price cannot be found after a sufficiently small period has elapsed from a known price is to assume that the security's return over that period is zero as the price is assumed to be constant.

For illiquid shares like CML and SKY were  $P_t$  may not change for 10 half-hour ticks or more at a time, the effect is that VaR can be zero as  $\sigma \approx 0$  for a lengthy period. This is not altogether inaccurate for as long as the portfolio is marked-to-market in the same way, then its return should also be zero for a similar period, meaning that the VaR estimate of loss would be close to the realized loss.

The effect of the above, however, is that when a trade actually does occur

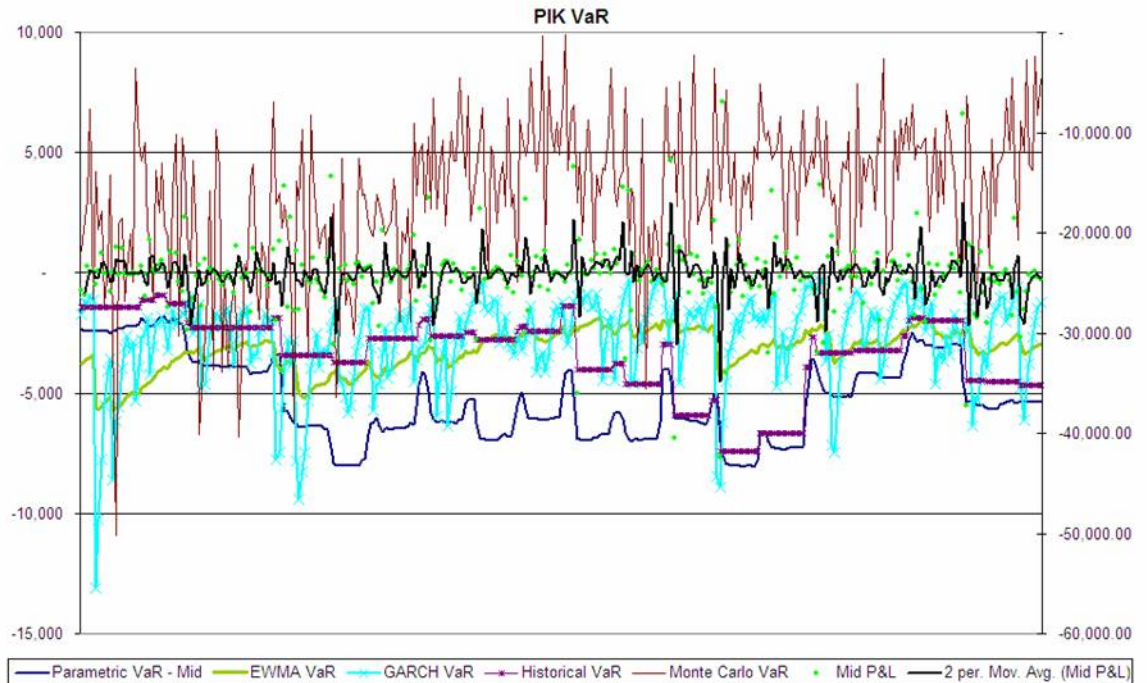


Figure 5.17: PIK Standard VaR Models

it tends to reflect the accumulated build up of price-sensitive information over the period and thus tends to be significantly different from the preceding price, rendering the return series and the resulting VaR estimate more volatile. This volatility tends to remain until the “anomalous” return has rolled out of the base.

The consequence of this volatility is seen clearly in the backtesting graphs of CML and SKY where the VaR estimate of loss is small but tends to peak abruptly and remain peaked for periods at a time. For these shares, the input parameters need a shorter memory and thus the GARCH model performs better.

As shown, the GARCH model as applied to CML and SKY displays much higher spikes than for PIK and AGL primarily because of the additional volatility in its estimate of  $\sigma$ . Figure 5.19, displays the slow update of the EWMA VaR more clearly than any of the previous graphs.

In general the standard VaR models do not match the realized P & L as closely as they do with PIK and AGL. This stems from the fact that these models do not address the special needs of illiquid stocks with their added

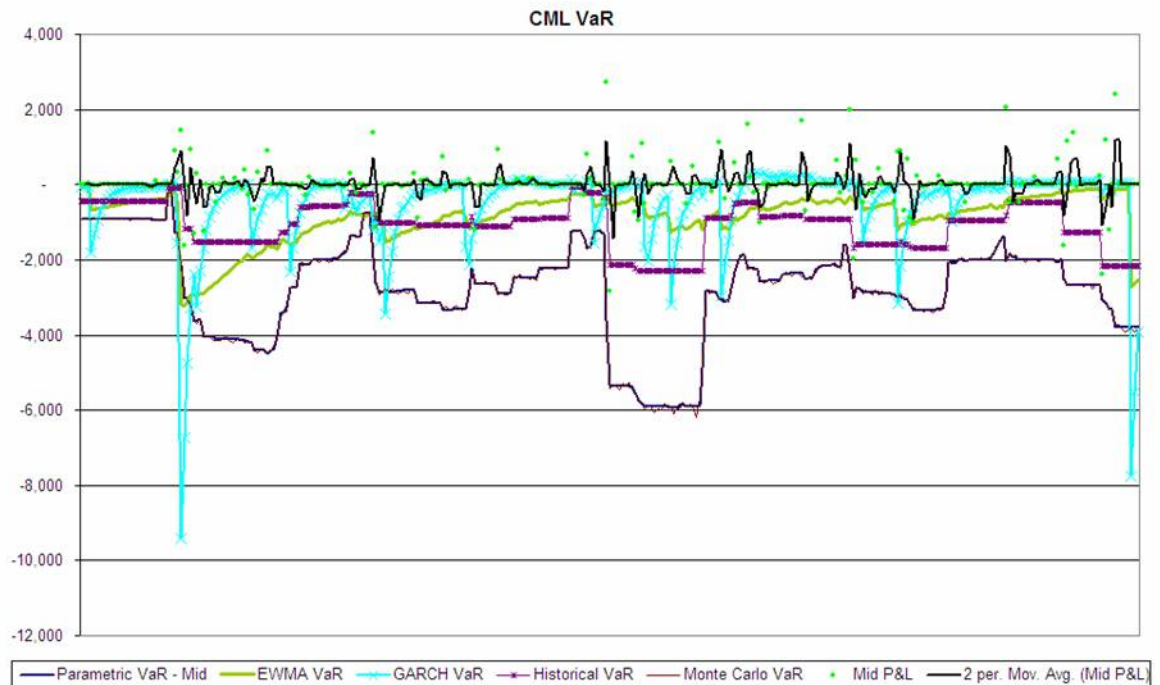


Figure 5.18: CML Standard VaR Models

return volatility and sudden price moves. All that they can do is to respond to changes in returns and retain these changes for different lengths of time. This seems largely inadequate for a measure which aims at forecasting loss and not merely perpetuating realized losses.

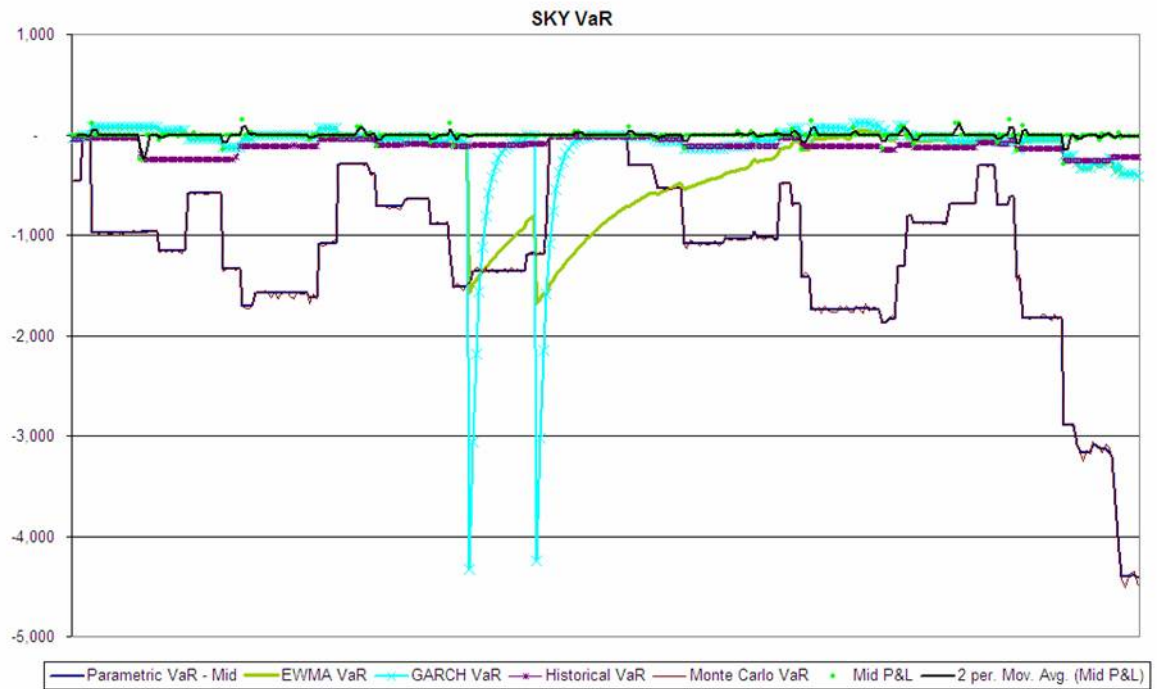


Figure 5.19: SKY Standard VaR Models

### Spread-Adjustment VaR Models

Of course, the standard VaR models cannot be discarded before they are more thoroughly compared against the alternatives. The graphs displayed in Figures 5.20, 5.21, 5.22 and 5.23 show the results of the spread-adjustment models.

As shown the spread-adjustment models seem more conservative than the standard VaR models. This is probably due to the addition of half the average relative spread which leaves the VaR estimate more negative and a function of both spread and return volatility. The VaR forecasts are, however, more stable and less sensitive to changes in the realized P & L than, for instance, the GARCH model. This is probably due to the muting effect brought about by the addition of the relative spread volatility. They display the same step-like functionality as noted in the previous models.

Similar to the standard VaR models, the spread-adjustment models respond to upward return volatility as well as downward volatility, meaning that VaR estimates tend to peak after large positive returns. The VaR models thus imply that returns have a higher probability of exhibiting losses after a period of positive returns.

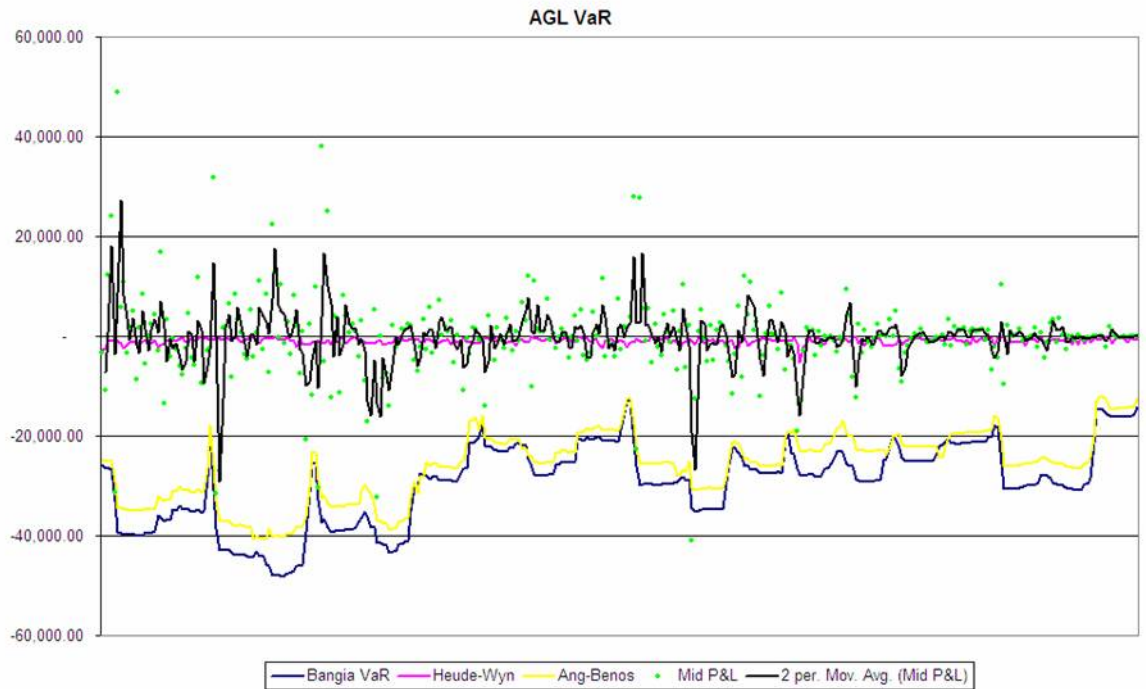


Figure 5.20: AGL Spread-Adjustment VaR Models

As is clear from Figure 5.20, the Bangia and the Angelidis models follow surprisingly similar trends. Indeed apart from the absolute difference in their levels they appear almost synchronized. This is somewhat surprising given the fact that their respective derivations are so dissimilar. Indeed the fact that the Angelidis model, based as it is on a structural model of the spread, comes so close to the Bangia model which uses the realized spread lends some validity to the structural model.

Based on the similarity of their derivation, a closer relationship between the François-Heude and Van Wynendaele and the Bangia model would have been expected. The backtesting results, however, make it clear that the attempt by François-Heude *et al* to model the spread without a pre-specified distribution and without the correlation assumption is ineffective. The François-Heude model is woefully inadequate at forecasting loss and displays poor coverage of the P & L distribution.



The reason for the failure seems to stem from the fact that the François-Heude model does not use standard parametric VaR as its focal point. Unlike the Bangia framework which appends a spread-adjustment to the parametric VaR to make it more conservative, thereby retaining the assumption that returns are normal, François-Heude *et al* apply the VaR methodology to a theoretical bid-price which is adjusted by half-the average relative spread. It assumes that the portfolio is marked-to-market at this theoretical bid price and bases its VaR on this assumption. Unfortunately in basing their VaR on this theoretical Bid they use the  $q$ th percentile of the mid-return distribution to get a  $q\%$  coverage for the  $VaR(q, V)$ .

Since the  $q$ th percentile of the mid-return distribution is itself a return, it is much lower than the corresponding  $q$ th percentile of the standard normal distribution used in the Bangia model and parametric VaR. The overriding effect is that the resulting VaR is far lower than standard parametric VaR. Thus while the François-Heude estimate of loss is more representative of the “standard” tick-to-tick loss experienced by the portfolio, it tends to be too low to provide sufficient coverage for the extreme losses which are not necessarily represented in the return set from which it derives its percentile. This is easily verified by observing the time trend of the François-Heude

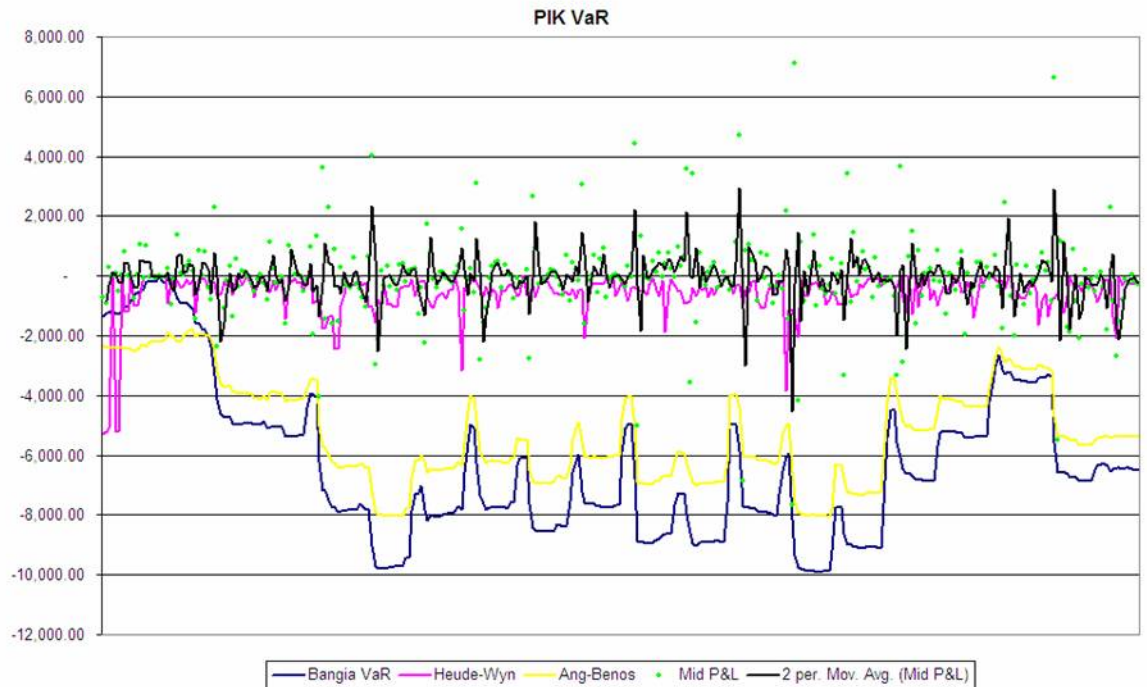


Figure 5.21: PIK Spread-Adjustment VaR Models

estimate in the backtesting graphs.

The François-Heude model does not really use the spread distribution at all. In contrast, the Angelidis model, like the Bangia model, retains the basic parametric VaR formulation and merely adjusts it for risks which are “ignored”. Hence their similarity to the standard VaR models and surprising accuracy.

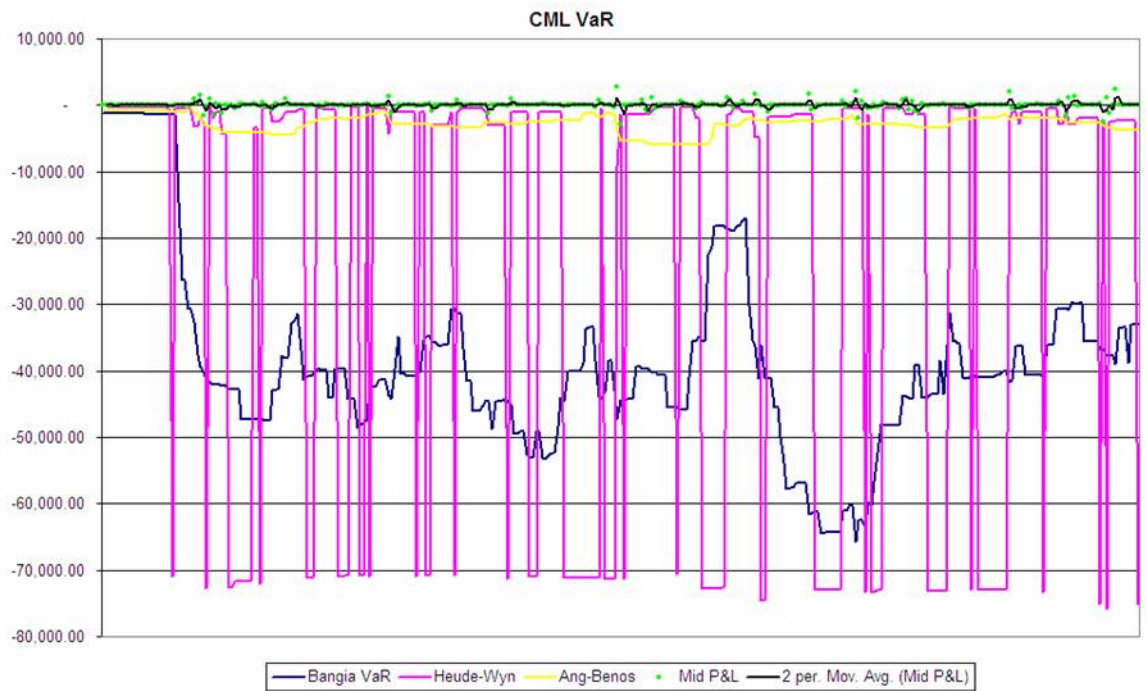


Figure 5.22: CML Spread-Adjustment VaR Models

A beneficial side effect of the use of the empirical mid-return distribution is that it tends to make the VaR estimate of loss much more sensitive to changes in the return. The VaR thus becomes more reactive and follows the change in the return distribution more closely. This is evident when comparing the François-Heude VaR series to the 2-point MA in Figures 5.20 and 5.21.

Apart from the Angelidis model, however, the remaining two spread-adjustment models do not seem to perform well in the case of the illiquid stocks. As is evident in Figures 5.22 and 5.23, the François-Heude VaR series displays volatility which clearly limits its usefulness, while the Bangia VaR’s estimate of loss is unduly large relative to the realized profit and loss. Indeed the relationship between the Bangia and Angelidis model seems to break-



down completely with the illiquid stocks. This is primarily caused by the dependence of these two models on the relative effective spread.

Since CML and SKY are illiquid stocks their relative effective spread is subject to sudden upward spikes brought about by the arrival of a Bid or Ask where previously there was none. Thus for many time ticks there are cases where  $A_t = 0$  or  $B_t = 0$  thereby raising the mid price,  $M_t$ , and inadvertently creating a relative spread series which is liable to sudden spikes. This renders VaR models which are heavily dependant on the spread more volatile and higher in absolute value.

Of course, the fact that only one side of a trade can be met at the time, implies that the stock is indeed illiquid so that the L-VaR should be made higher. However, it must be made higher in a way that remains realistic and sensible.

Figures 5.24 and 5.25 show the relative effective spread for CML and SKY against the relevant spread-adjustment models and the 30-point realized relative spread moving average. It clearly shows that the reason for the François-Heude VaR's inaccuracy is its inordinate dependence on the spread.

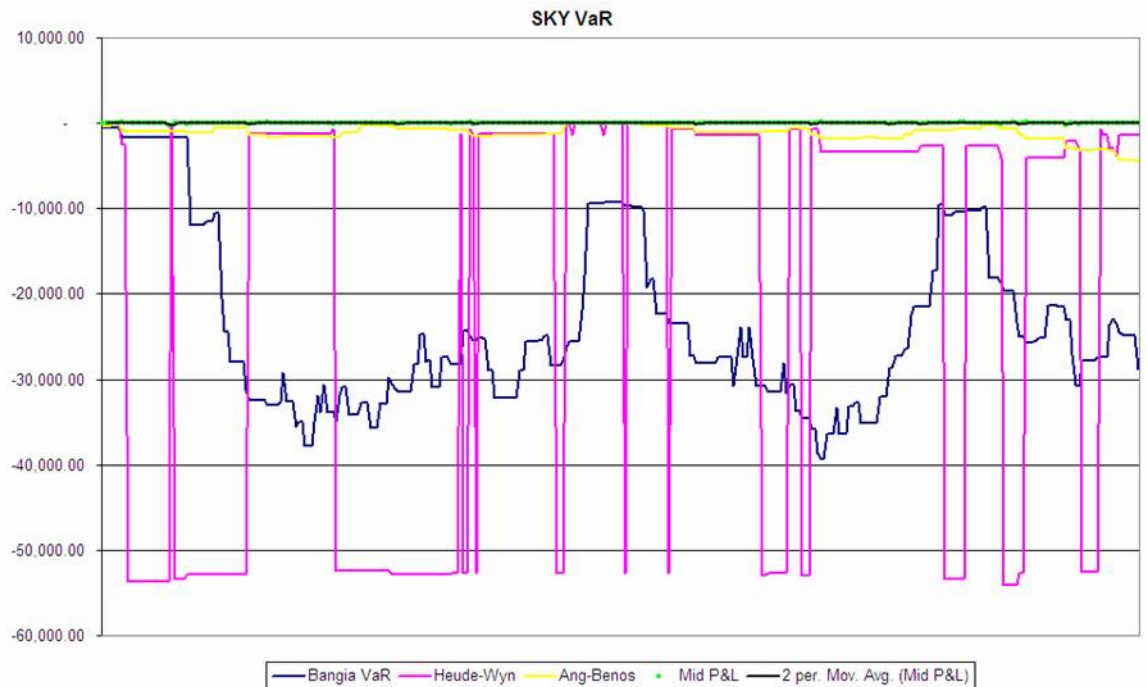


Figure 5.23: SKY Spread-Adjustment VaR Models

Similarly the Bangia VaR, driven as it is by an average relative spread which is made higher by sudden upward spikes, produces higher than necessary estimates of likely loss.

The Angelidis VaR model produces estimates of loss based on a spread-based structural model which achieves the needs of being more conservative than standard parametric VaR while remaining stable and practical. In this model, the addition of the liquidity element is seamless and although the liquidity-component of the total VaR is small (in the region of only 1% difference to standard parametric VaR but certainly bigger for the illiquid stocks) it does have an impact. The fact that this impact occurs without rendering the model too dependant on changes in the relative effective spread is significant.

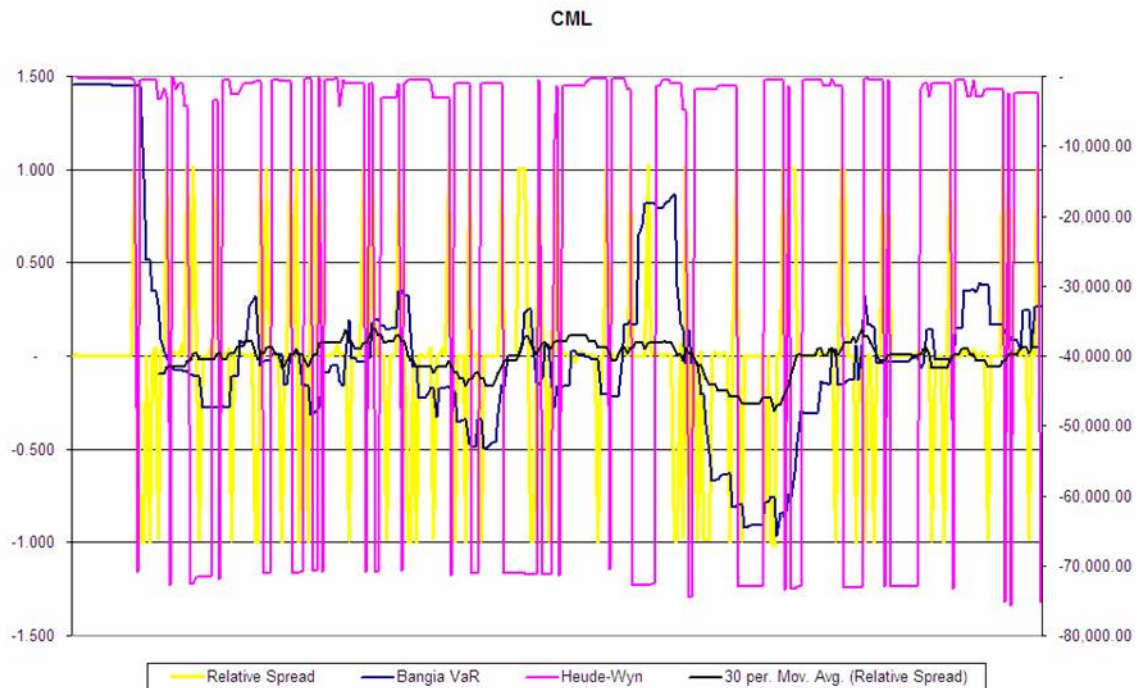


Figure 5.24: CML Spread and VaR models

### Parametric-Adjustment VaR Models

Although the Angelidis L-VaR manages to incorporate the spread-based aspects of liquidity risk without causing too many difficulties, the fact that even the Bangia model may be too dependant on changes in the spread begs the question whether discarding the spread and accounting for liquidity-risk

using a parametric adjustment would be easier. Figures 5.26, 5.27, 5.28 and 5.29 showcase the parametric adjustment models which aim at achieving just this.

In all of the figures, the Al Janabi VaR series is far higher than any of the other models. It is plotted against the right, secondary axis for greater clarity as its estimate of loss is approximately 6 times larger than the other VaR models. This inordinate conservatism stems from the scaling adjustment implicit to the Al Janabi model. As noted in Section 4.5.2, the Al Janabi scales parametric VaR by a factor such that  $\sqrt{\frac{(2t+1)(t+1)}{6t}} \rightarrow \infty$  as  $t \rightarrow \infty$ .

The model effectively assumes then that as the time to liquidation of a position increases, the risks associated with it increases multiplicatively independent of the position size or the trading strategy. This is clearly not sensible. The simple example of a long equity position for instance is a case in point <sup>4</sup>. The backtesting graphs validate the fact that the model is not

<sup>4</sup>A long equity position, no matter how large, always has its maximal loss limited to its current market value or initial cost. From the investor's perspective loss cannot increase without bound

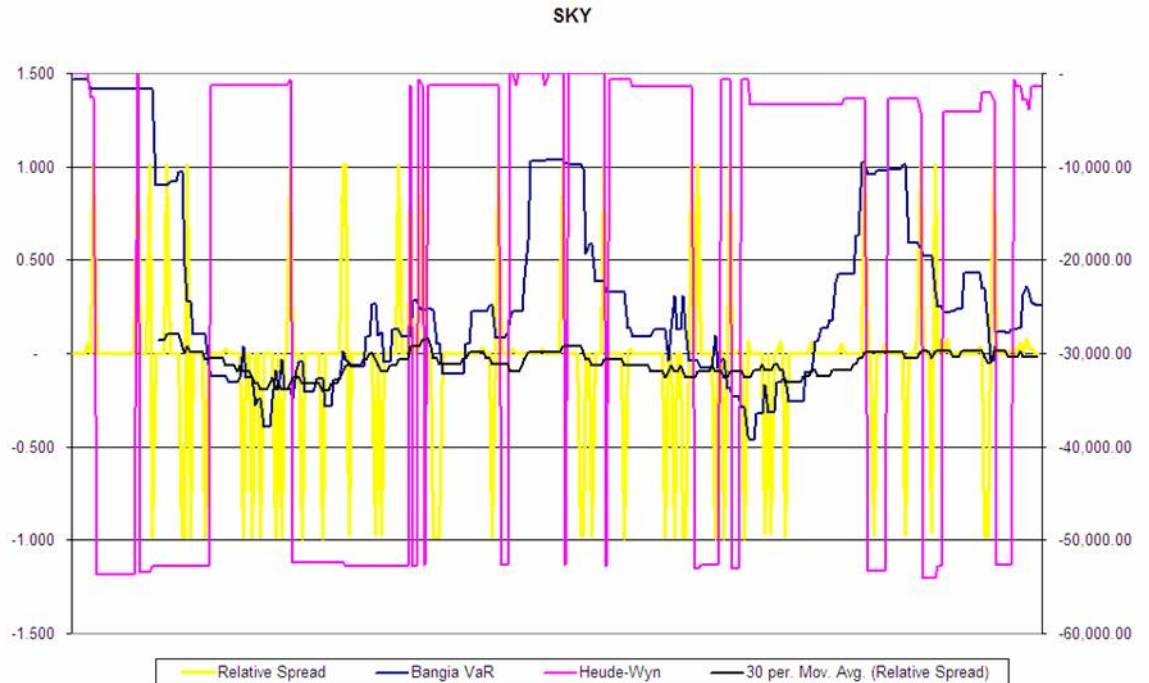


Figure 5.25: SKY Spread and VaR Models

robust and show that its use for capital adequacy purposes would lead to a prohibitively high cost of capital.

Apart from the failing of the Al Janabi model, however, the parametric-adjustment models fair quite well. They display good coverage of the realized P & L distribution and seem to adjust well to changes in a share's liquidity, exhibiting none of the added volatility that the spread-adjustment models do in the case of CML and SKY. This is displayed in Figures 5.28 and 5.29.

All of the models, with the exception of Berkowitz, display the same step-functionality that is indicative of the parametric VaR models. The Berkowitz VaR series displays a far more smooth trend with localised peaks which seems unrelated to the other VaR models. This probably arises as a consequence of the models use of actual portfolio flow data as opposed to mid returns. The use of flow data incubates the model to some extent from the added mid-return and spread volatility which is characteristic of illiquid shares.

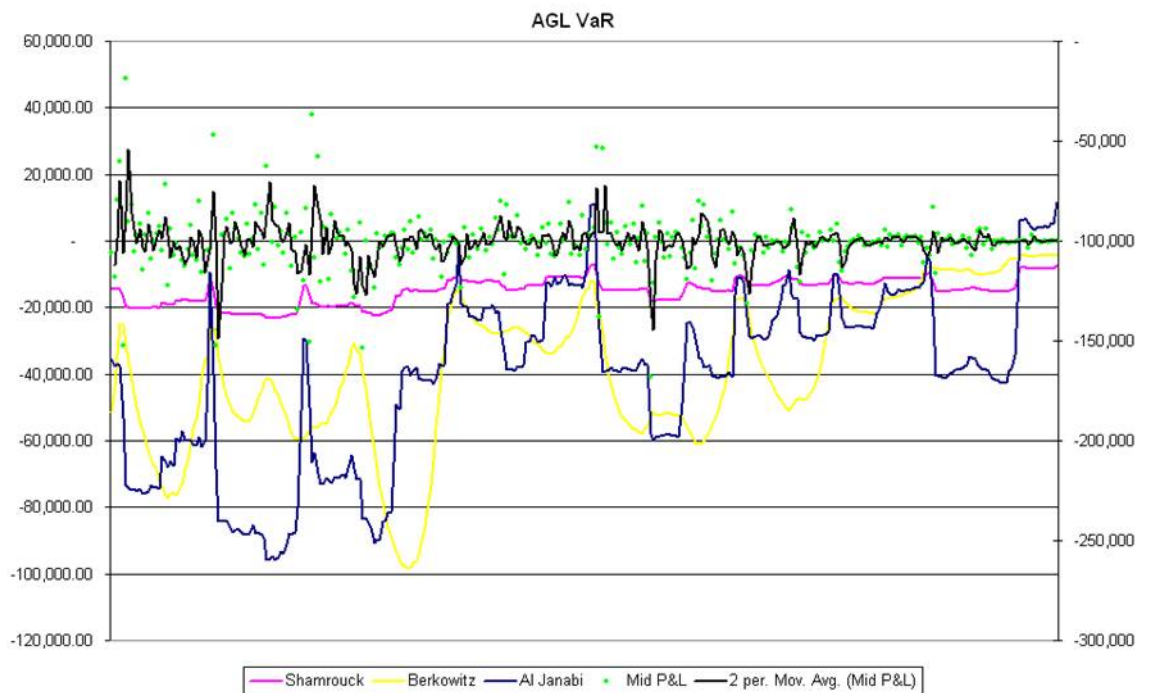


Figure 5.26: AGL Parametric-Adjustment VaR Models

## Optimal Liquidation Strategy VaR Models

The OLS models represent the most theoretically complete and rigorous liquidity risk metrics. They aim at capturing the totality of liquidity risk and many of the models, like the Hisata & Yamai model, do so very elegantly with very neat formulations.

The cost, unfortunately, of this added sophistication is that the models are difficult to implement and require the estimation of parameters which are extremely changeable and largely nebulous. This complicates their implementation and makes their widespread use particularly sensitive to the model's accuracy – the additional modelling burden must be compensated by forecasts which are more accurate.

As shown by the backtesting graphs in Figures 5.30, 5.31, 5.32 and 5.33 this is not necessarily achieved. Based on the graphs it does not seem clear that the additional rigour inherent to these models leads to more accurate estimates. While the loss forecasts are most certainly more conservative than many of the preceding VaR models, they are also a lot more volatile, exhibiting extreme changes which are not always associated with changes in

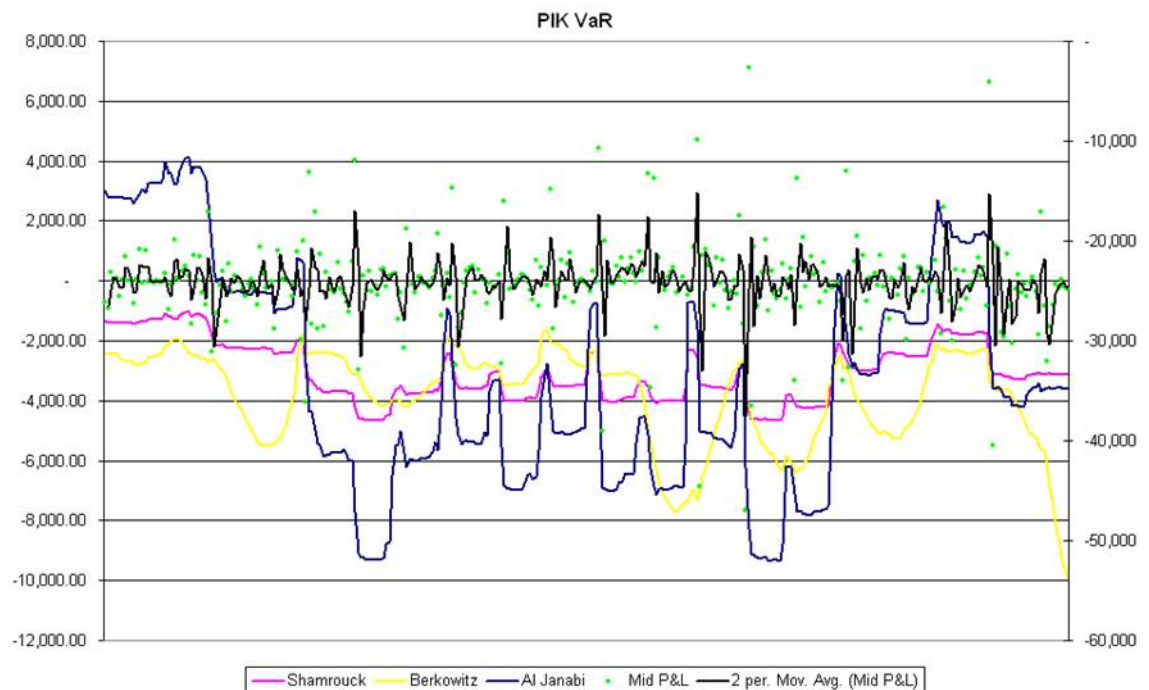


Figure 5.27: PIK Parametric-Adjustment VaR Models

the realized portfolio loss. This becomes more problematic with the illiquid stocks.

With AGL for instance, the VaR series seems to follow the moves in the 2-point MA closely but with some degree of overreaction to large changes in the realized profit or loss. The estimates of loss are somewhat indicative of the losses experienced by the portfolio. However as shown in Figures 5.31, 5.32 and 5.33, this does not remain true. As the stocks become more illiquid, the model's estimate of loss becomes more conservative relative to the realized loss. This is particularly true of the Almgren & Chriss model which displays the most drastic trends and forecasts.

Both the Almgren & Chriss and the Hisata & Yamai models are highly dependant on the estimate of the expected proceeds from liquidation,  $\mathbb{E}(x)$ . Increases in  $\mathbb{E}(x)$  translate into direct increases in the VaR forecast. Thus any sensitivity which  $\mathbb{E}(x)$  has to changes in the parameter set  $(\gamma, \eta, \varepsilon)$  carries over directly into the VaR model. This is particularly problematic given the difficulty in estimating these parameters and the lack of guidance given by the literature into their estimation.

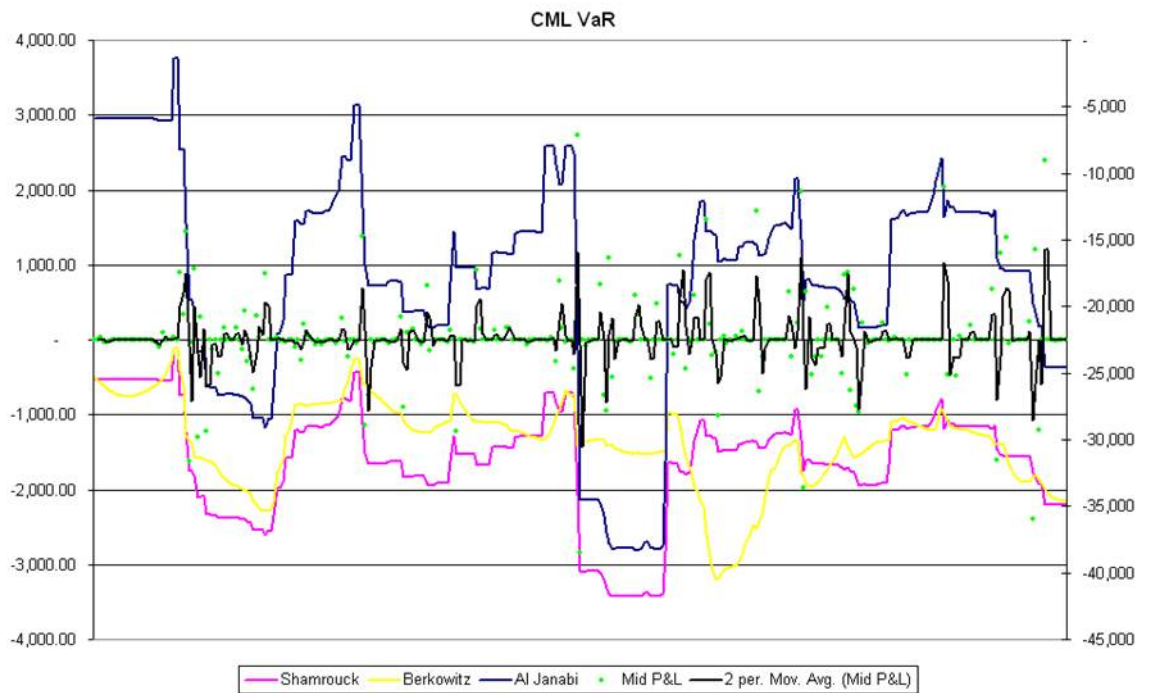


Figure 5.28: CML Parametric-Adjustment VaR Models



By their very nature, the price impact parameters are highly dependant on prices and associated traded volumes and are prone to extreme changes. This limits the length of time that any particular estimate can be considered accurate, further complicating their estimation.

Moreover since no particular form, as shown in Figure 4.4, is known to be the most accurate for their estimation, trying to come close to the parameters' true value is an onerous task, which compounds the model risk inherent to OLS VaR.

The importance which the price impacts play in the OLS models warrants some analysis into their levels. Figure 5.34 shows the average and standard deviation of the various price impact parameters used in the models. The values for  $\gamma$  are plotted on the right axis.

What is clear from Figure 5.34 is that both the permanent price impact, as measured by  $\eta$ , and the temporary price impact,  $\gamma$ , is more volatile and, on average, takes on marginally higher values for the liquid stocks.

The additional volatility of the liquid stocks' price impacts may be due to the

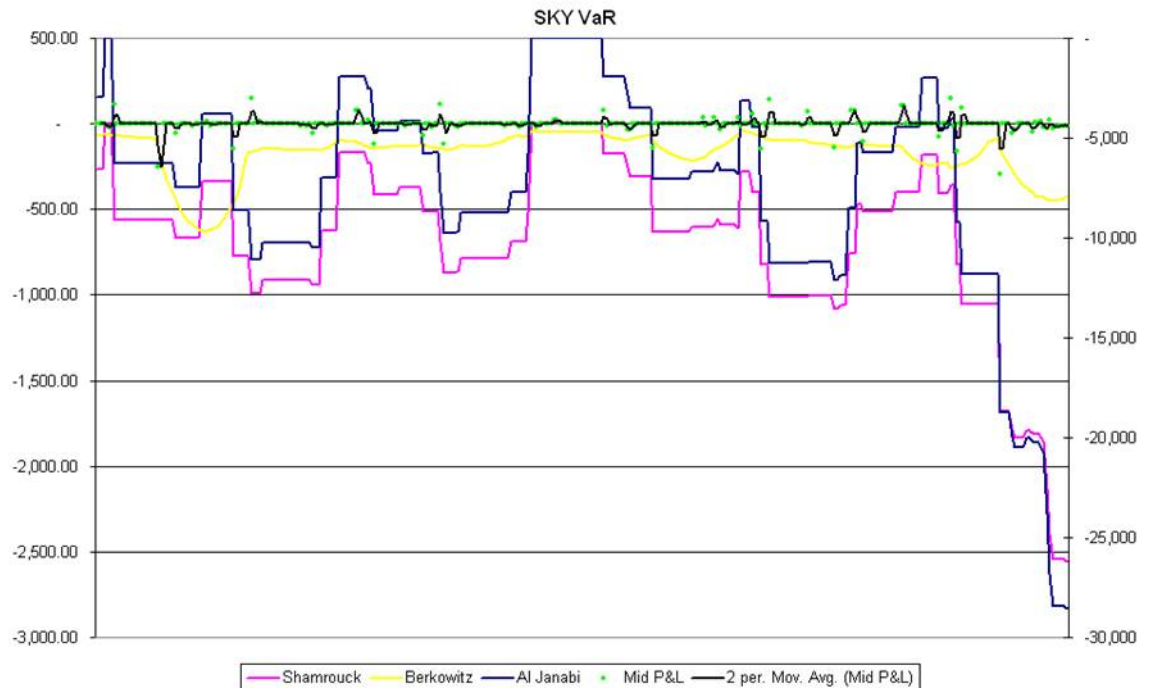


Figure 5.29: SKY Parametric-Adjustment VaR Models

fact that liquid stocks respond more quickly to changes in trade which are perceived to carry price-sensitive information. This seems reasonable given the fact that these stocks tend to be the most highly traded and thus closely watched. Expectations tend to be more homogeneous for the liquid stocks over time, making their price discovery process more supportive. Thus as trades enter the market, other traders quickly adjust to the new order flow in a way which would reduce their risk. This has the effect of increasing both forms of price impacts and rendering the impacts more volatile.

Interestingly, in line with the behaviour of the spread-adjustment models,  $\varepsilon$  is on average higher for the illiquid stocks than for the liquid stocks.

Since the Hisata & Yamai model is so directly dependant on  $\eta$ , much of its variability then stems from changes in this parameter. This is also the reason behind the VaR model's higher estimate of loss for the liquid stocks. Unfortunately no similar reason can be given for the Almgren & Chriss VaR forecasts which do not bear as simple a relationship to the price impact parameters.

Overall the OLS models yield inordinately high forecasts of loss which ex-

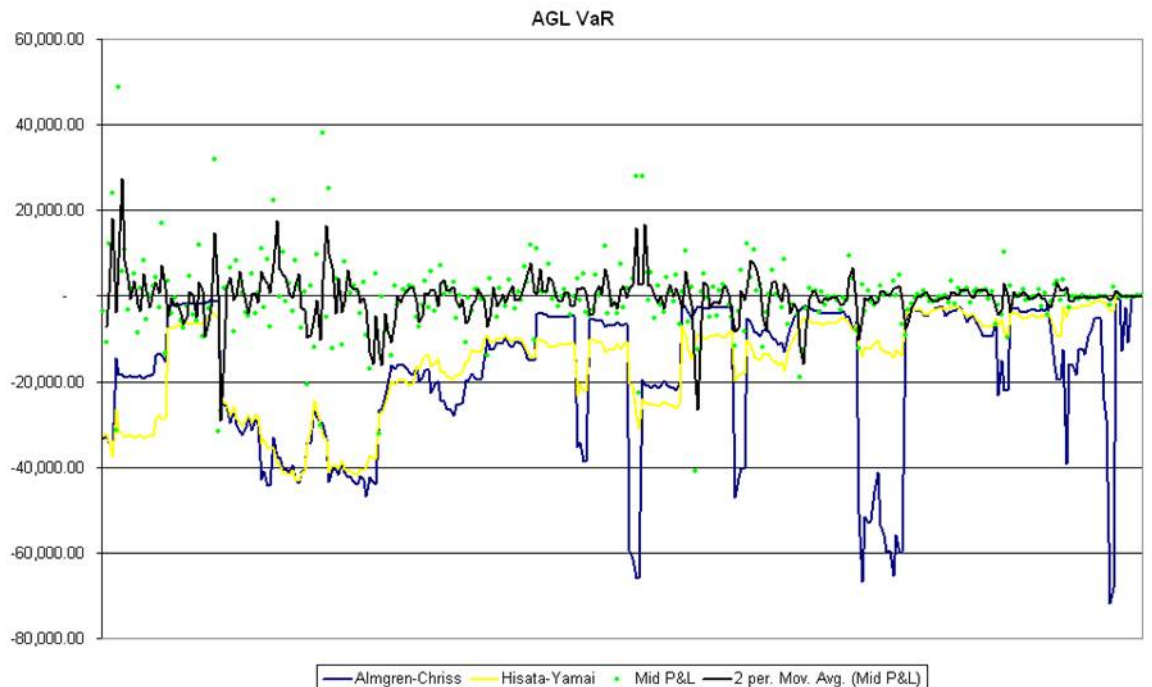


Figure 5.30: AGL Optimal Liquidation Strategy VaR Models



hibit very conservative coverage of the P & L distribution. Although this is not as bad as the Al Janabi model, the additional sophistication of these models had created hopes that they would be far more accurate. The back-testing graphs, however, belie the fact that these models cannot be easily implemented and that the additional accuracy gained in fitting a complicated liquidity VaR model is often completely offset by the inaccuracy resulting from a lack of sufficient data to estimate parameters properly. This is particularly true of the illiquid stocks where data is scarce and often discontinuous.

## Model Comparisons

The VaR models must be compared not only within their groups but across groups so as to highlight the differences in their estimates for differences in liquidity. Figures 5.35, 5.36, 5.37 and 5.38, display the promising L-VaR models for each of the stocks considered.

The Figures make it clear that while no single VaR model provides a consistently higher forecast of loss, the standard parametric VaR's forecast is

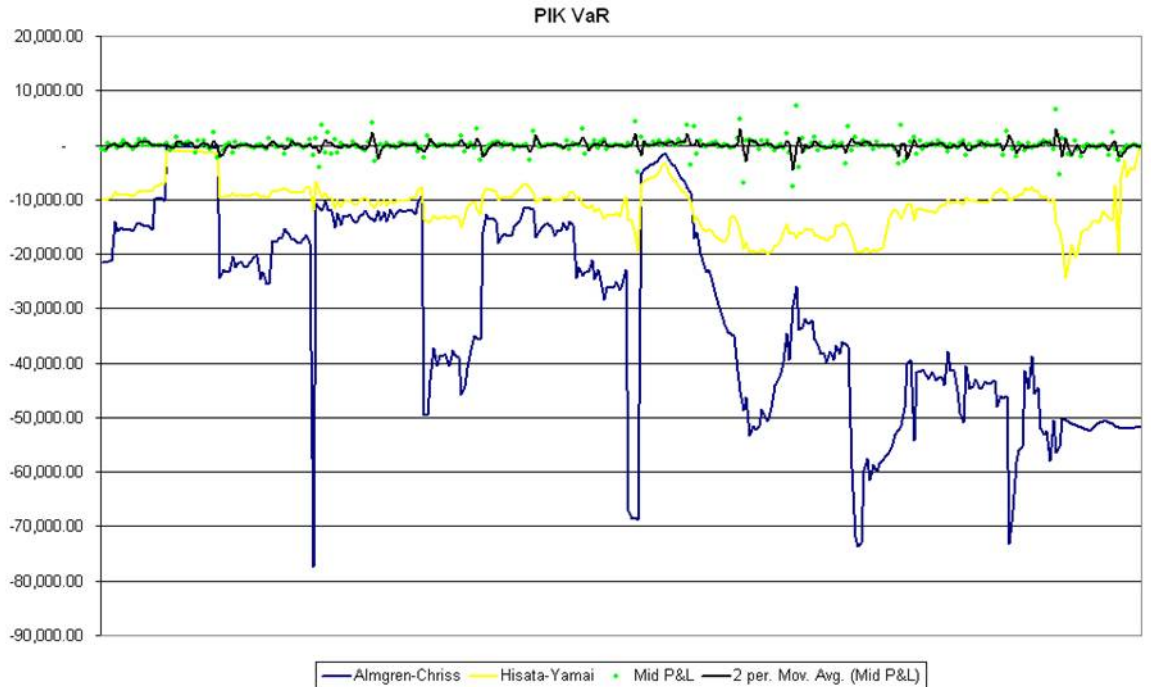


Figure 5.31: PIK Optimal Liquidation Strategy VaR Models

generally higher than that of the other models. This is true for most of the stocks but changes in the case of the illiquid shares where the Bangia VaR and the Hisata & Yamai models provide more extreme forecasts.

As shown in Figure 5.35, generally many of the VaR models follow similar trends but with different outlier behaviour. This is expected as many of the L-VaR models take standard parametric VaR as their base. The models also display similar coverage of the P & L distribution with the parametric VaR and the Berkowitz models showing the greatest conservatism. This changes, however, once more illiquid shares are considered.

In the case of the illiquid shares, it is the Hisata & Yamai and the Bangia models which dominate, producing estimates of loss which are characterized by larger spikes and far higher levels than the other models. Figures 5.36, 5.37 and 5.38 make this self-evident, showing the stark difference between the models' behaviour for AGL.

Although both of the models must be admonished for being somewhat too sensitive to decreases in liquidity, to its credit, the Hisata model's forecasts remain reasonable and sensitive to changes in realized profits and losses.

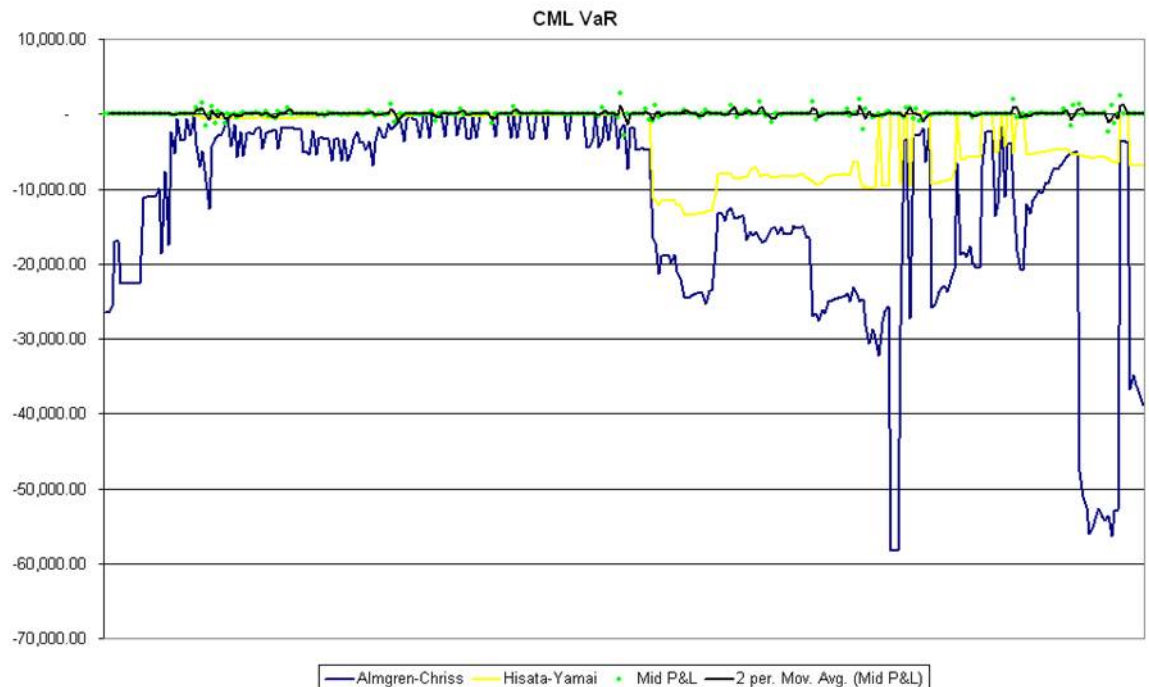


Figure 5.32: CML Optimal Liquidation Strategy VaR Models

The Bangia model's dependence, however, on the relative effective spread and the problems this poses for illiquid stocks (as discussed earlier) renders its forecasts prohibitively inaccurate.

The Hisata & Yamai and Bangia models are easy scapegoats, however. Since their forecasts are so inordinately high in the case of the illiquid stocks, they seem to be more inaccurate. Closer examination of the trends across the graphs show that really all of the models provide forecasts which become less indicative of the actual portfolio loss as liquidity decreases. As shown in Figure 5.38, while the portfolio loss is never more than a few hundred Rand, the VaR models provide forecasts of loss which are in the order of thousands of Rands. This, of course, raises questions regarding the models' representativeness in times of stress when liquidity losses become even more prominent.

Overall, however, the Berkowitz VaR and the Historical VaR models seem to be the most flexible in terms of adjusting to changes in liquidity. While their absolute accuracy cannot be guaranteed until the models undergo statistical testing, at least on the basis of the trends in backtesting graphs they seem to produce forecasts which are both representative of realized loss and not

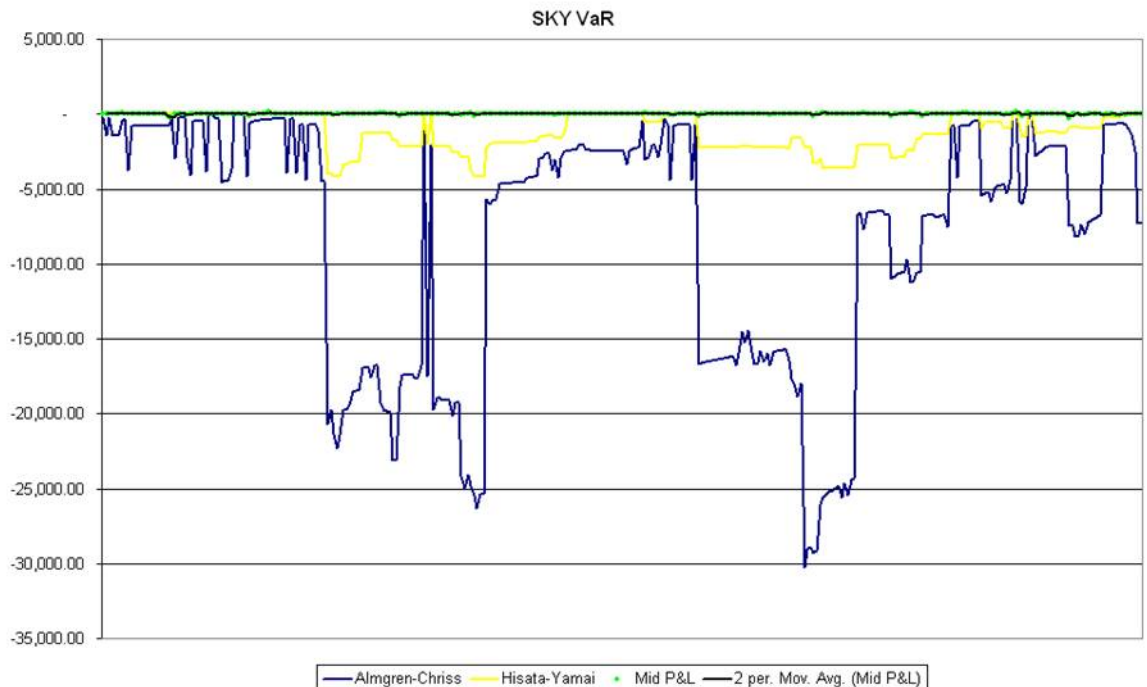


Figure 5.33: SKY Optimal Liquidation Strategy VaR Models

subject to unnecessary swings.

Although the VaR models cannot be completely described by their means and variances, the examination of these descriptive statistics does impart some useful, general information on the differences between the models.

Figures 5.39 and 5.41 display the average and variance of the forecast values assumed by each of the VaR models for each of the stocks considered. Given the wide range which these values assume between the liquid and illiquid stocks, Figures 5.40 displays the average for CML and SKY alone. The average of the realized loss is displayed on the right-hand axis and is represented by the scatter point in each graph.

Generally the average and variance graphs support the points made earlier regarding the relative size and volatility of the VaR models. As shown the Al Janabi model provides by far the most conservative estimate of loss in the case of the liquid stocks, but in the case of CML and SKY where the relative effective spread assumes greater significance, it is somewhat eclipsed by the Bangia and François-Heude VaR models. Both of these models have the property of becoming far more conservative when liquidity is constrained.

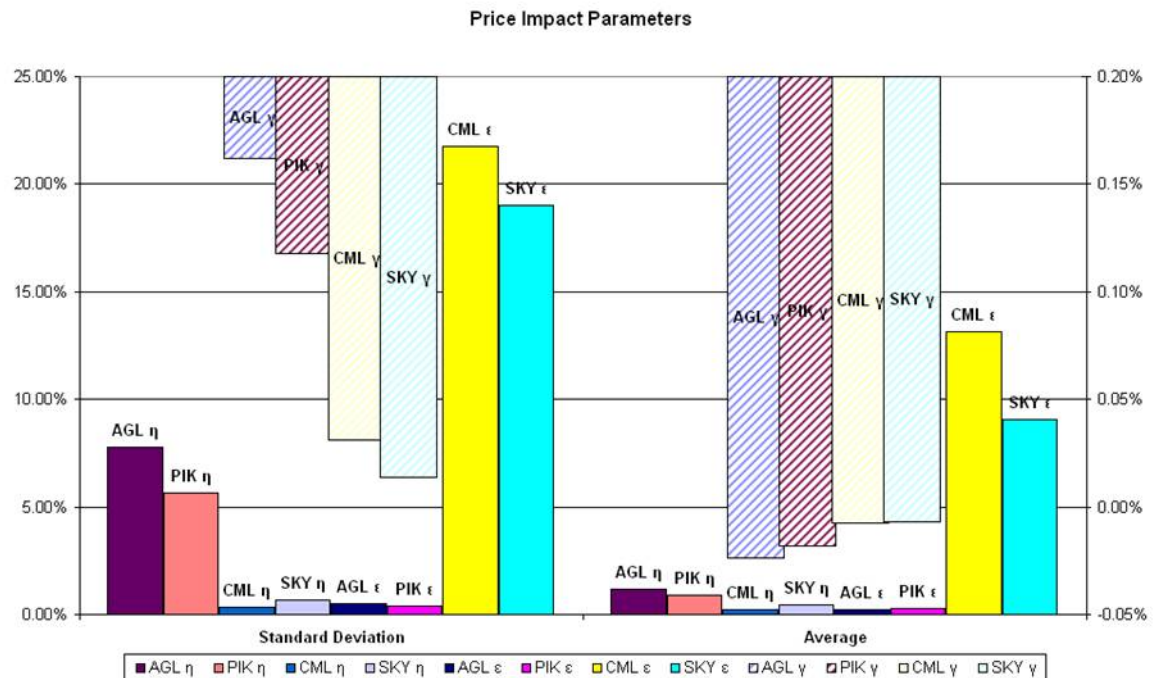


Figure 5.34: Average & Standard Deviation of Price Impact Parameters

They also become more volatile on a relative basis in this instance.

The other spread-adjustment model, the Angelidis model, does not share this property and remains relatively stable despite changes in liquidity. It provides forecasts which are far less volatile than any of the other models and which comes closest on average to the values assumed by the parametric VaR and Monte Carlo VaR models.

Similarly, the Shamroukh and the Berkowitz models provide much less volatile forecasts, which are on average closer to the average forecast across the models. The Berkowitz VaR does, however, display anomalous behaviour in the case of AGL where its average is made higher by a few sudden spikes.

The Figures give support for the conclusion that models which account for price impacts like the Hisata & Yamai VaR, and even the Almgren & Chriss VaR, respond to changes in liquidity more reasonably than the simplistic Bangia Var and the François-Heude variation. This is particularly evident for the illiquid shares where both spread-adjustment models display undue volatility. Although the volatility associated with the VaR forecasts are extreme given that they are based on highly volatile, half-hour tick returns, the

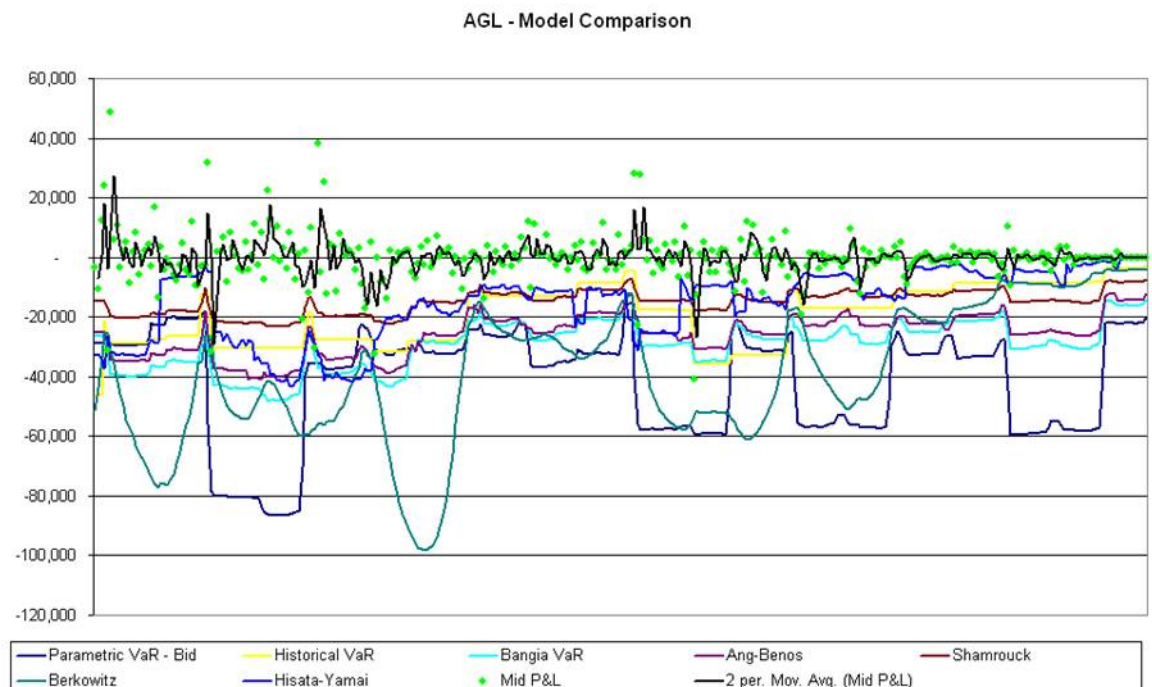


Figure 5.35: AGL VaR Model Comparison

differences in the models are what is important and highlight the potential behaviour of the models if instruments were indeed as volatile as presented.

Overall, however, Figure 5.39 gives no way of consistently ranking the models across differing levels of liquidity. The models tend to respond to changes in a stock's liquidity in a way which is not obvious and is masked by the lower absolute level of cash flows attributable to the illiquid stocks.

Despite this, a trend does at least emerge regarding the difference in the parametric VaR when based on the bid, ask, mid or trade price. Parametric VaR forecasts based on the Bid and the Ask tend to produce more volatile and higher forecasts of loss than those based on the trade or mid prices. Generally, forecasts based on the Bid are higher than those based on the Ask (but not always more volatile).

The difference in the standard VaR for differences in the valuation price highlights the importance which different mark-to-market regimes can have on portfolio risk. It hints at the quote which opens Chapter 1 – at times when the market is particularly illiquid, standard valuations and risk model estimates can be driven so completely off sync that different underlying

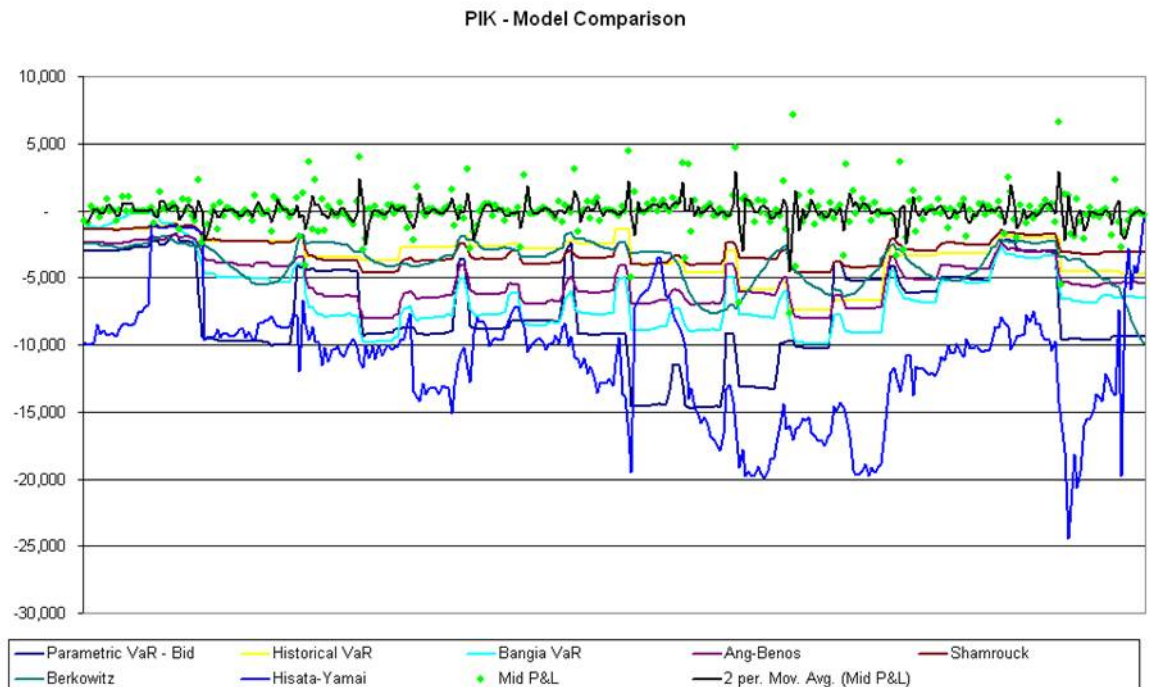


Figure 5.36: PIK VaR Model Comparison



valuations are needed to assess risk. In certain grave circumstances neither the Bid nor the Ask nor the Mid price are adequate representations of what an investor could get upon liquidation. Risk models must account for this.

## Sensitivity Analysis

As discussed in Section 3.6, models need to be relatively insensitive to changes in any single parameter so as to prevent estimation error from detracting from the model's performance.

Analysing a model's sensitivity to its parameters is an important component of model analysis and identifies the key risk areas inherent to the model's design. Unless identified prior to a model's implementation, undue sensitivities to any particular parameter can confound existing weaknesses in the model and leave forecasts vulnerable to extreme error.

Sensitivity analysis also helps in scenario analysis as it allows users to quickly gauge the effect of a particular change in market conditions or portfolio composition on total portfolio risk.

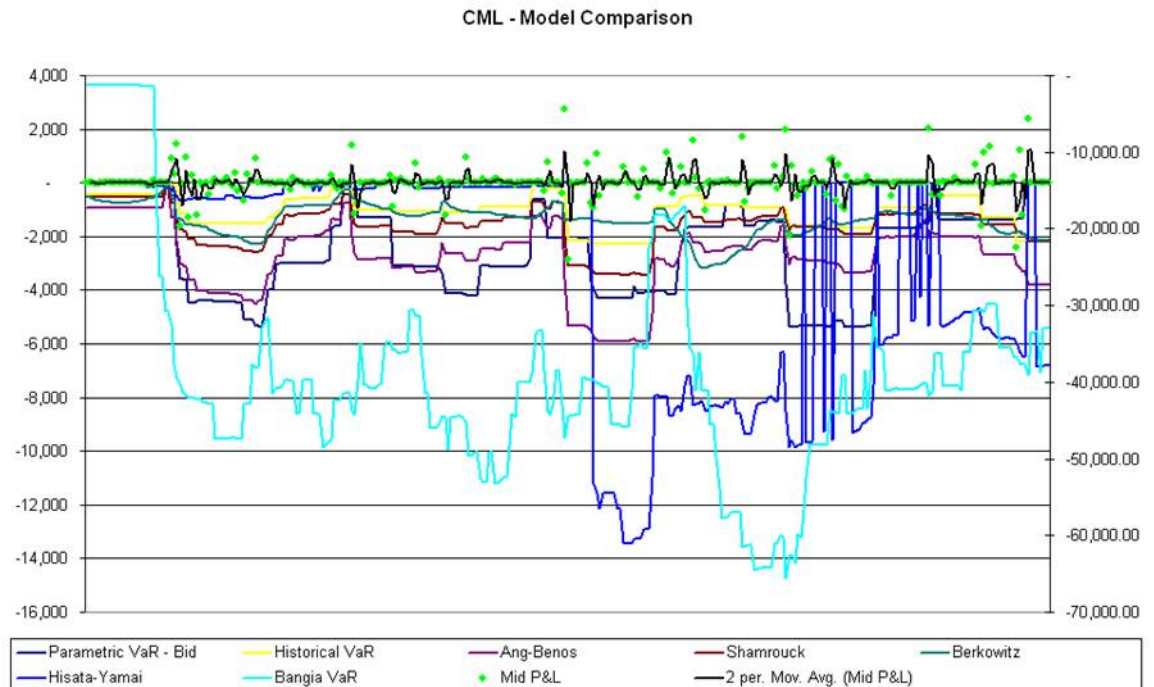


Figure 5.37: CML VaR Model Comparison

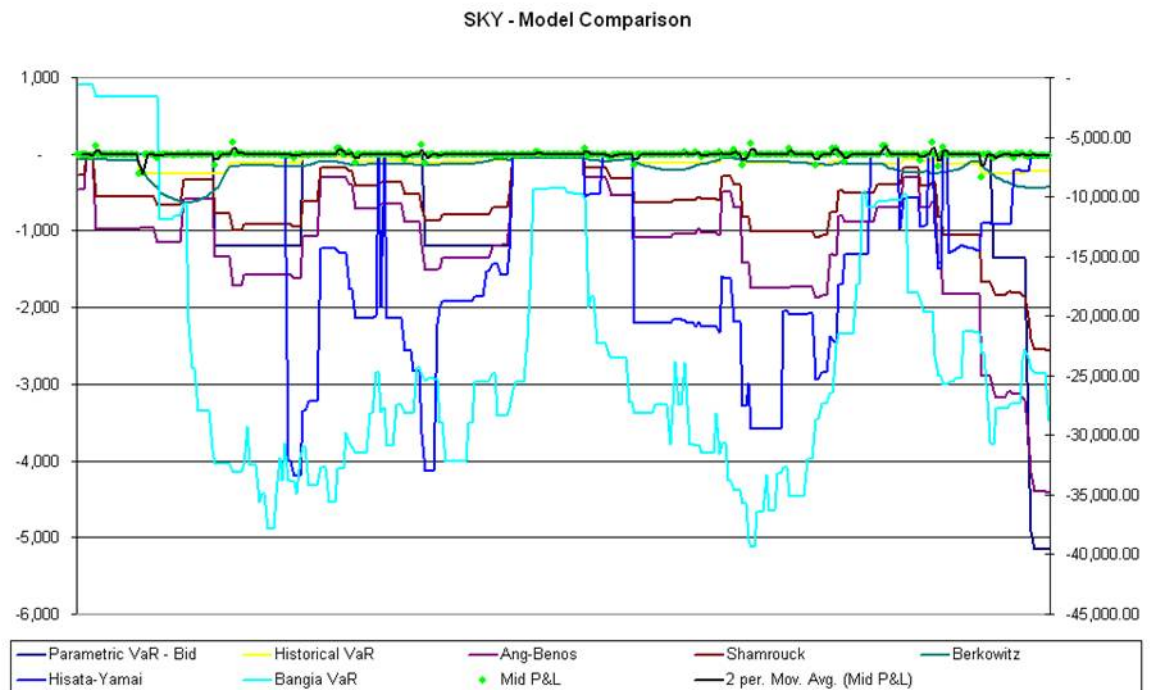


Figure 5.38: SKY VaR Model Comparison

Although many of the L-VaR models implemented do not share input parameters, it remains useful to compare the models' sensitivity in one table. Tables 5.12 and 5.13 display the % change in the L-VaR estimate of loss for a +100%, 50% and -100% change in the respective parameter input.



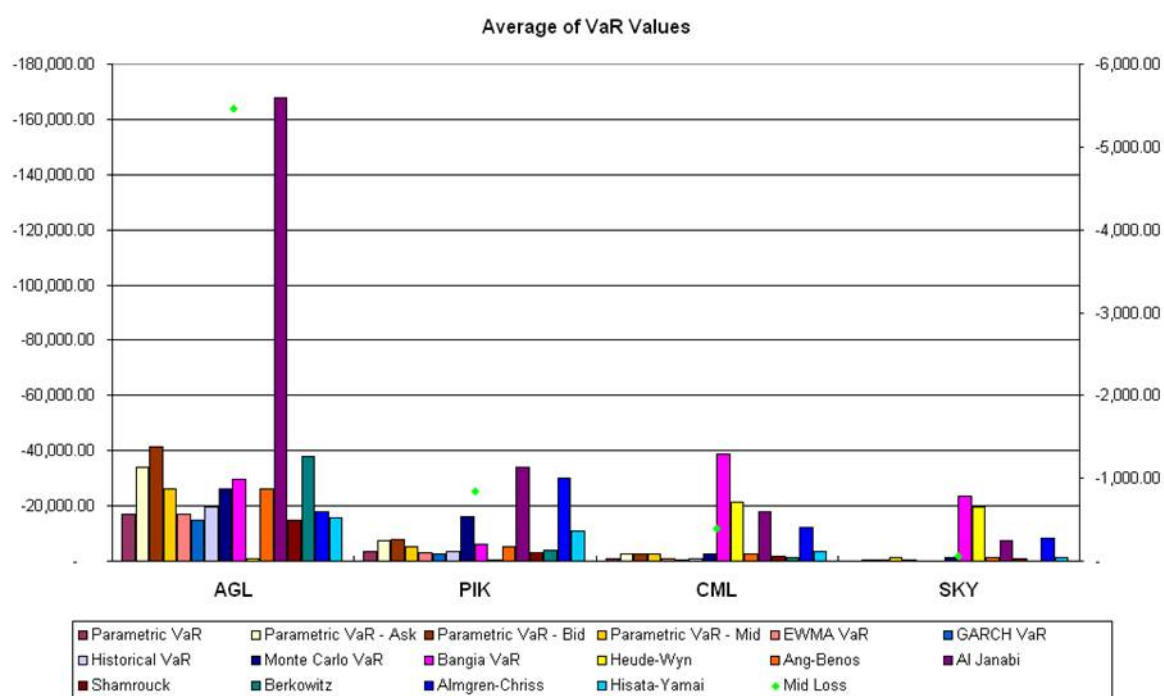


Figure 5.39: Average VaR Forecasts across Models

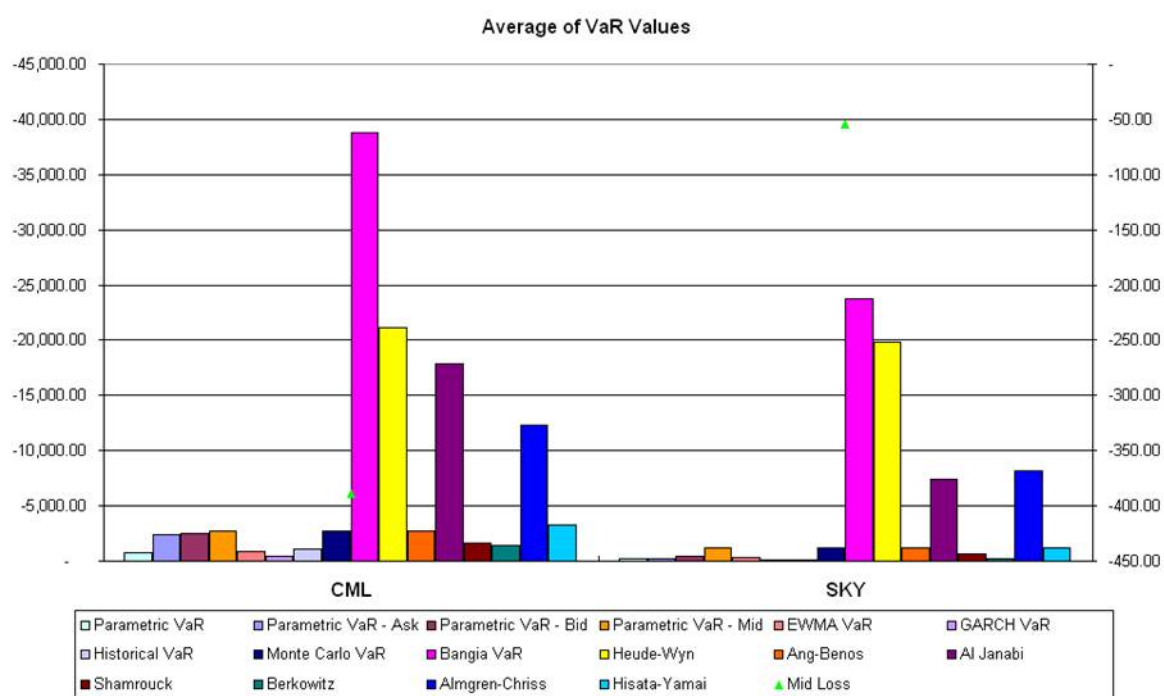


Figure 5.40: Average VaR Forecasts across Models

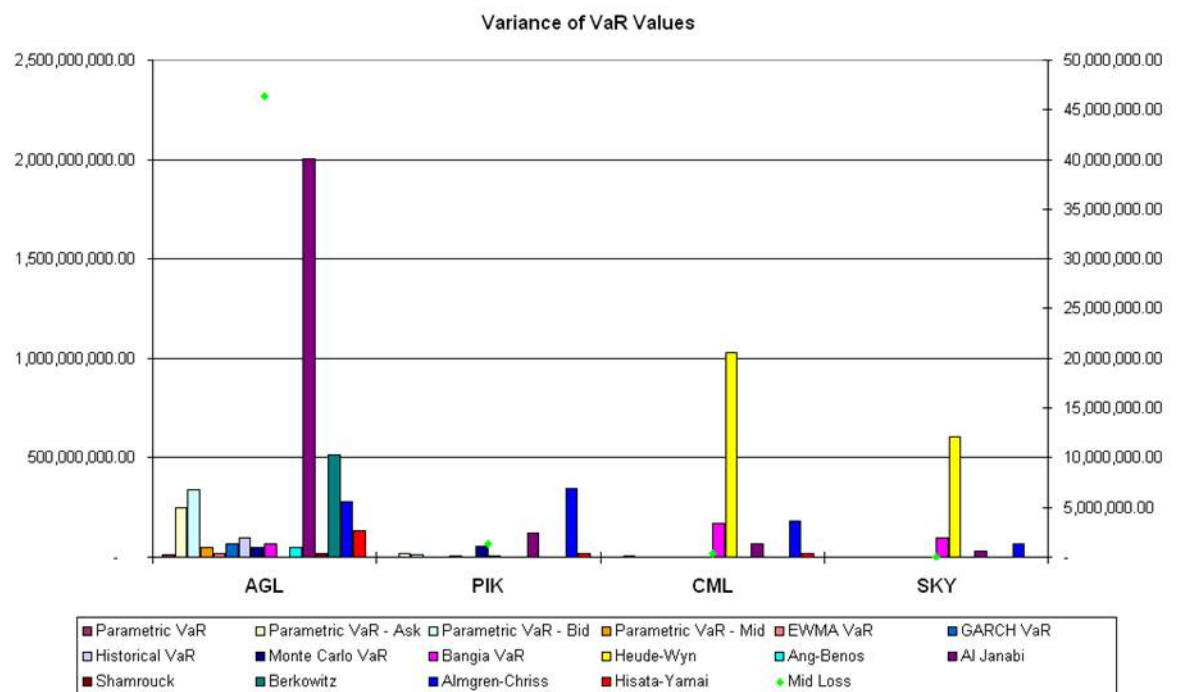


Figure 5.41: Variance of VaR Forecasts across Models

	Input	% Change	Parametric	EWMA	GARCH	Bangia	Heude-Wyn	Ang-Benos	Al Janabi	Shanroukh	Berkowitz	Almgren-Chriss	Hisata-Yamai
Mid Return Average	$\mu$	100%	-5%	-7%	-4%	-4%	0%	-5%	-5%	-5%	0%	0%	0%
		50%	-2%	-3%	-2%	-2%	0%	-2%	-2%	-2%	0%	0%	0%
		-100%	5%	7%	4%	4%	0%	5%	5%	5%	0%	0%	0%
Mid Return Volatility	$\sigma$	100%	103%	37%	17%	97%	-21%	103%	103%	103%	94%	111%	59%
		50%	52%	20%	9%	49%	-11%	52%	52%	52%	47%	52%	31%
		-100%	-105%	-61%	-21%	-99%	21%	-105%	-105%	-105%	-94%	-53%	-100%
EWMA VaR	$\lambda$	100%	0%	-7%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	-7%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	77%	0%	0%	0%	0%	0%	0%	0%	0%	0%
GARCH Inputs	$\alpha$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
	$\beta$	100%	0%	0%	13%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	7%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	-15%	0%	0%	0%	0%	0%	0%	0%	0%
Rel Spread Average	$\hat{S}$	100%	0%	0%	0%	4%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	2%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	-4%	0%	0%	0%	0%	0%	0%	0%
Rel Spread Spot	$S$	100%	0%	0%	0%	0%	121%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	61%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	-121%	0%	0%	0%	0%	0%	0%
Rel Spread Volatility	$\sigma$	100%	0%	0%	0%	1%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	1%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	-1%	0%	0%	0%	0%	0%	0%	0%
qth Percentile of Mid Return Dbn	$\alpha$	100%	0%	0%	0%	0%	-21%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	-11%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	21%	0%	0%	0%	0%	0%	0%
Bangia Fat-Tail Factor	$\theta$	100%	0%	0%	0%	97%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	49%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	-99%	0%	0%	0%	0%	0%	0%	0%

Table 5.12: Sensitivity Analysis 1

	Input	% Change	Parametric	EWMA	GARCH	Bangia	Heude-Wyn	Ang-Benos	Al Janabi	Shanroukh	Berkowitz	Almgren-Chriss	Hisata-Yamai
Ang & Benos - Information asymmetry Factor	$\theta$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Ang & Benos - Dependant Trade Factor	$\kappa$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Ang & Benos - Independent Trade Factor	$\Phi$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
qth Percentile of Volume trade Dbn	$V_i$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Time to liquidation	$t$	100%	0%	0%	0%	0%	0%	0%	41%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	22%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	-37%	0%	0%	0%	0%
Number of trade dates to liquidation	$n$	100%	0%	0%	0%	0%	0%	0%	0%	41%	0%	28%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	22%	0%	17%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	606%	0%	-53%	0%
Average Historical Amount Traded	$q_t$	100%	0%	0%	0%	0%	0%	0%	0%	0%	6%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	3%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	-6%	0%	0%
Average Change in Historical Trade Price	$x_t$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Berkowitz Trade Impact	$\theta$	100%	0%	0%	0%	0%	0%	0%	0%	0%	6%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	3%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	-6%	0%	0%
Permanent Price Impact	$\gamma$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	47%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	24%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	-47%	0%
Temporary Price Impact	$\eta$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	5%	26%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	3%	14%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	-51%	-100%
Fixed Exogenous Costs	$\varepsilon$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Trade time-steps	$\tau$	100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	28%	0%
		50%	0%	0%	0%	0%	0%	0%	0%	0%	0%	16%	0%
		-100%	0%	0%	0%	0%	0%	0%	0%	0%	0%	-24%	0%

Table 5.13: Sensitivity Analysis 2

As can be seen, all of the models, with the exception of the Berkowitz VaR, are sensitive to changes in the mid return  $\mu$ . Similar changes in the parameter tend to induce similar effects across the models, with all of them generally increasing by 5% for a 100% decrease in  $\mu$ . All of the models also seem to react symmetrically to changes in the parameter, exhibiting the same increase/decrease in absolute value for a 100% decrease/increase in  $\mu$ .

Interestingly, the VaR models are far more sensitive to changes in the return volatility  $\sigma$  than perhaps any of their other input parameters. With the exception of the François-Heude, EWMA and GARCH models, increases in  $\sigma$  lead to direct increases in the VaR number. This is particularly true of the spread-adjustment and price-impact models where return volatility seems to play an extremely important role.

The great sensitivity which the models have to changes in  $\sigma$  supports the trend in the literature to improve VaR by improving the estimates of volatility. It could be possible, based on the sensitivity analysis, then to drastically improve the forecasts of some of the less accurate models like the price-impact models, the Bangia model, etc. with the use of better volatility estimates. To the extent that volatility accounts for illiquidity because of the discontinuous “jumps” evident in the prices of illiquid stocks, such an adjustment could improve the L-VaR models. This, of course, needs further research.

Introducing a EWMA or GARCH methodology into the  $\sigma$  estimation of the other L-VaR models may be a step in this direction but would also leave the models dependant on the additional parameters. Although *RiskMetrics<sup>TM</sup>* advises a figure of  $\lambda = 94\%$  for their EWMA VaR, it is evident that the VaR is quite sensitive to changes in the parameter in ways which are difficult to predict. While a 100% increase in  $\lambda$  induces only a 7% decrease in the VaR number, a similar 100% decrease induces a much larger 77% increase in the VaR. The sensitivity also seems to change depending on the base level of  $\lambda$  used in the VaR calculation.

Changes in  $\alpha$  are directly offset by changes in  $\gamma$  and vice versa, leaving the GARCH VaR independent to changes in these parameters. Changes in  $\beta$  do have a small impact, however.

Given the differences in their methodologies, the various spread-adjustment models share very few inputs with the Angelidis model, based as it is on a structural model of the spread, which requires the most inputs. Luckily, however, many of these parameters are easy to estimate and remain relatively stable. The fact that the model is also extremely insensitive to

changes in these unique parameters – the sensitivities are mostly 0 – greatly reduces model risk.

The Francoise-Heude model displays an inordinately high sensitivity to changes in the relative effective spread. Changes in this market variable translate almost directly into changes in the VaR number with some additional scaling. The sensitivity table provides firm evidence for the earlier argument that the reason behind the model’s undue volatility in the case of the illiquid stocks was the great variability in their spread.

The Bangia VaR’s reduced accuracy in the case of the illiquid stocks cannot, however, be explained by increased spread volatility. As shown, the model’s sensitivity to both the average spread and the current spread is very small. The model, however, does display large changes in response to changes in its fat tail factor  $\theta$ . This makes the use of the number as estimated in the original paper questionable. In order to fulfil the Bangia model’s potential, it seems necessary to estimate  $\theta$  accurately, irrespective of the difficulty involved in doing this. The likely reason behind the failure of the Bangia model with CML and SKY is an inappropriate fat-tail correction.

The parametric-adjustment models like Shamroukh and Al Janabi offer little in the way of a sensitivity investigation. Although they are sensitive to changes in the investment horizon  $t$  and the number of trade points  $n$ , these parameters are pre-determined and hardly change. They thus offer little modelling risk.

The Berkowitz VaR, like the Angelidis model, displays greater accuracy (as shown by the backtesting graphs) and very low sensitivity to changes in its parameter inputs. These are desirable qualities in a model.

As expected, the price-impact models display large sensitivities to changes in their parameter inputs. While the models are relatively insensitive to changes in the endogenous fixed cost variable (which echoes the discussion in Section 2.1 that exogenous price impacts are roughly only one third of the total liquidity cost),  $\varepsilon$ , their forecasts are prone to large changes in response to changes in the endogenous price impact variables. The Almgren & Chriss model, in particular seems highly dependant on both  $\gamma$  and  $\eta$  while the Hisata & Yamai model, having  $\eta$  as its only significant variable, is only sensitive to  $\eta$ .

The high sensitivity of the complex price-impact models to changes in their inputs highlights an essential problem – while more sophisticated and complete models may seem, on the basis of theory, to be more accurate they can be prone to implementation problems. Although the two price-impact

models are laudably complete, more research into their practical implementation is crucial if the models are to gain wider use. Not only must more work be done into the price impact estimation process but the nature of the relationship between the models and their price impact parameters needs to be investigated. This will go a long way to making the models more practicable.

## Statistical Backtesting

Although the above analysis of the VaR model's time trends and general behaviour provides important comparative information, the ultimate test of a VaR model lies in determining how closely its forecasts match realized loss. This is the preserve of VaR Backtesting techniques which aim at assessing how accurate a specific VaR model is at forecasting.

The general idea behind backtesting techniques is that on average a portfolio  $V$  should display losses which breach its VaR,  $VaR(\lambda, V)$ , only  $1 - \lambda\%$  of the time for the VaR methodology to be considered accurate. A VaR model can thus easily be assessed by counting the percentage of "breaches". Specific VaR-related hypothesis tests do, however, exist to make testing more thorough. The details of these are supplied in Appendix C.

While the backtesting of VaR has received a great deal of attention over time and a plethora of methodologies have been put forward by the literature to make backtesting more accurate, only the main statistical tests for Conditional Coverage and Independence are applied in this analysis.

Figure 5.42 shows the percentage of realized portfolio losses which fall below the VaR estimate of loss over time. Since all the VaR models are implemented at a 99% level, all of the models should display coverage close to this threshold, represented by the dotted black line.

The coverage graph makes the failure of the François-Heude VaR for the liquid stocks immediately apparent. The model seems woefully inadequate and systematically underestimates portfolio loss for stocks where the spread is less volatile and a smaller component of aggregate liquidity risk. For AGL and PIK this model only captures on average 65% of the realized portfolio losses. This improves to 95% for CML and SKY.

Interestingly, Figure 5.42 shows that while the parametric models are relatively good for the liquid stocks, as liquidity declines they become gradually worse. This is true of the parametric VaR based on the bid, ask and trade



prices. Only the parametric VaR based on the mid price remains consistently accurate, while the EWMA and GARCH regimes seem to trail off severely.

Contrastingly the Historical VaR and the Monte Carlo VaR models provide stable and accurate forecasts with coverage levels which remain around 97%. Generally the Monte Carlo VaR seems to be more conservative than either the parametric Mid VaR or the Historical VaR. While the Historic VaR does not perform too poorly, its coverage levels are not exactly at the threshold. This implies that the model may have too many breaches to be consistently accurate, particularly during trend-breaking market moves.

The coverage table hides the fact that some of the models, like Al Janabi, consistently over-estimate portfolio loss. The Al Janabi, Bangia, Angelidis, Monte Carlo and even the parametric Mid VaR have cases where they are higher than all of the portfolio's realized losses. While this makes the models more conservative and prudent, it does not make them more accurate.

The Shamroukh, Berkowitz and Angelidis models provide the most stable coverage which is consistently close to the 99% threshold.

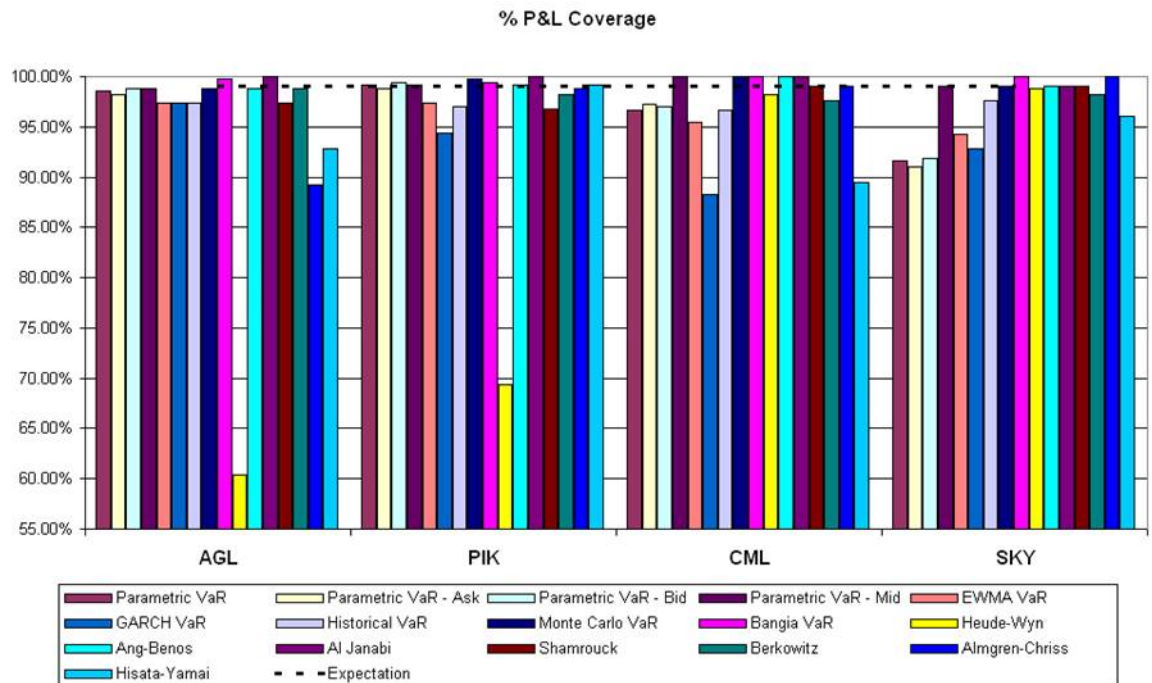


Figure 5.42: Percentage Profit & Loss Coverage by VaR Models

Disappointingly, despite the lauded sophistication and rigour of the Hisata & Yamai and Almgren & Chriss models, they display coverage which averages around 95%. The models seem to underestimate loss. This is surprising given the dominant upward swings shown in the backtesting graphs.

Due to scale effects, the backtesting graphs emphasise the large swings in the models' forecasts and hide the fact that their estimates of loss are accurate. Overall the models provide forecasts which are close to the actual portfolio loss but which are prone to exaggerated swings in response to changes in their input parameters. These swings occur at the incorrect time thus the forecasts seem too small in response to large realized losses and too large when they exhibit an exaggerate response. Closer examination of Figures 5.37 and 5.38 support this – the models produce estimates which are on average only slightly larger than the other models (Figure 5.39 supports this too) but which display inaccuracies at specific localities.

Although the reason for this behaviour may be diverse, it most probably lies with the fact that parameter estimation for these sophisticated price-impact models is complicated by the data “holes” that characterize the illiquid stocks. It echoes the earlier discussion that the additional accuracy offered by these models can be largely offset by estimation error and lack of data.

As noted, the coverage test only highlights a model's flaws if it tends to underestimate portfolio risk. It gives no indication really if a portfolio tends to overestimate risk. Although a model which underestimates risk is more troubling than a model which is too prudent, where capital adequacy issues play a role both issues need to be considered.

In addition to this, the coverage table offers no clear definition of “good enough” coverage. Based on the table alone it is unclear whether a model with 95% coverage is just as good as a model with 92% coverage. A more robust hypothesis test is needed to complete the picture.

Fortunately VaR backtesting techniques have found ways to address both of these issues. While the specific theoretical underpinnings of the tests are discussed in Appendix C, the literature largely applies two tests in assessing a VaR model: the Kupiec Conditional Coverage test and the Christoffersen Independence test.

The first test provides a framework with which to judge whether a model provides “good enough” coverage of the realized losses it tries to forecast, while the second test determines if a VaR model systematically under or over-estimates risk. The tests are combined in a joint test of conditional coverage and independence, developed by Christoffersen [54].

Stock	Statistic	Param VaR	Param VaR Ask	Param VaR Bid	Param VaR Mid	EWMA VaR	GARCH VaR
AGL	% of P&L Coverage	98.51%	98.21%	98.81%	98.81%	97.32%	97.32%
AGL	Unconditional Cov Stat - LRuc	0.7	1.7	0.12	0.12	6.55	6.55
AGL	Independence Stat - LRind	-3.15	2.32	1.34	1.54	-2.58	-2.88
AGL	Joint Test - LRuc + LRind	-2.45	4.02	1.45	1.66	3.97	3.67
PIK	% of P&L Coverage	99.11%	98.81%	99.4%	99.11%	97.32%	94.35%
PIK	Unconditional Cov Stat - LRuc	0.04	0.12	0.65	0.04	6.55	<b>35.3</b>
PIK	Independence Stat - LRind	0.23	1.52	0.19	0.28	1.22	-9.13
PIK	Joint Test - LRuc + LRind	0.27	1.64	0.84	0.32	7.77	<b>26.18</b>
CML	% of P&L Coverage	96.68%	97.28%	96.98%	100%	95.47%	88.22%
CML	Unconditional Cov Stat - LRuc	<b>11.22</b>	<b>6.72</b>	<b>8.87</b>	<b>6.65</b>	<b>22.38</b>	<b>125.05</b>
CML	Independence Stat - LRind	4.33	-10.8	-8.84	0	-6.58	-19.97
CML	Joint Test - LRuc + LRind	<b>15.55</b>	-4.07	0.03	6.65	<b>15.79</b>	<b>105.08</b>
SKY	% of P&L Coverage	91.62%	91.02%	91.92%	99.1%	94.31%	92.81%
SKY	Unconditional Cov Stat - LRuc	<b>71.64</b>	<b>80.6</b>	<b>67.27</b>	0.04	<b>35.5</b>	<b>54.66</b>
SKY	Independence Stat - LRind	-22.79	-6.49	-2.73	-2.94	-8.63	-7.83
SKY	Joint Test - LRuc + LRind	<b>48.84</b>	<b>74.11</b>	<b>64.54</b>	-2.9	<b>26.86</b>	<b>46.83</b>

Table 5.14: Statistical Backtesting Results 1

Stock	Statistic	Historical VaR	Monte Carlo VaR	Bangia VaR	Heude-Wyn	Ang-Benos	Al Janabi
AGL	% of P&L Coverage	97.32%	98.81%	99.7%	60.42%	98.81%	100%
AGL	Unconditional Cov Stat - LRuc	6.55	0.12	2.31	<b>777.95</b>	0.12	<b>6.75</b>
AGL	Independence Stat - LRind	3.5	1.54	0.11	-78.86	1.54	<b>FAIL</b>
AGL	Joint Test - LRuc + LRind	<b>10.05</b>	1.66	2.43	<b>699.1</b>	1.66	<b>FAIL</b>
PIK	% of P&L Coverage	97.02%	99.7%	99.4%	69.35%	99.11%	100%
PIK	Unconditional Cov Stat - LRuc	<b>8.67</b>	2.31	0.65	<b>539.19</b>	0.04	<b>6.75</b>
PIK	Independence Stat - LRind	3.49	0.14	0.75	-68.02	0.28	<b>FAIL</b>
PIK	Joint Test - LRuc + LRind	<b>12.15</b>	2.46	1.4	<b>471.16</b>	0.32	<b>FAIL</b>
CML	% of P&L Coverage	96.68%	100%	100%	98.19%	100%	100%
CML	Unconditional Cov Stat - LRuc	<b>11.22</b>	<b>6.65</b>	<b>6.65</b>	1.78	<b>6.65</b>	<b>6.65</b>
CML	Independence Stat - LRind	3.81	0	0	1.51	0	<b>FAIL</b>
CML	Joint Test - LRuc + LRind	<b>15.03</b>	6.65	6.65	3.29	6.65	<b>FAIL</b>
SKY	% of P&L Coverage	97.6%	99.1%	100%	98.8%	99.1%	99.1%
SKY	Unconditional Cov Stat - LRuc	2.95	0.04	<b>6.71</b>	0.12	0.04	0.04
SKY	Independence Stat - LRind	2.53	-2.94	<b>FAIL</b>	-3.05	-2.94	-2.94
SKY	Joint Test - LRuc + LRind	5.47	-2.9	<b>FAIL</b>	-2.93	-2.9	-2.9

Table 5.15: Statistical Backtesting Results 2

All of the above statistical tests are applied to the VaR and realized profit and loss series. The independence test is implemented with the assistance of VBA code which is displayed in Appendix D.

Tables 5.14, 5.15 and 5.16 display the results of both the separate and the joint hypothesis tests. Where the relevant test statistic is greater than its 1% critical, the test statistic is highlighted. In this case the null hypothesis that the VaR model is accurate (where accuracy is defined by the test itself) can be rejected at the 1% level. In certain instances, the models have displayed such poor results that the test statistic fails. These models are severely inadequate and are indicated by “FAIL”.

The testing tables reflect the earlier contention that standard parametric VaR models perform very well for the liquid stocks but tend to become progressively worse as liquidity falls. Generally all of them, including the EWMA and GARCH methodologies, fail in the case of CML and SKY. Only the parametric VaR based on the Mid, the Historic VaR and the Monte Carlo VaR pass in the case of SKY. This highlights the tenuous relationship

Stock	Statistic	Shanroukh	Berkowitz	Almgren-Chriss	Hisata-Yamai
AGL	% of P&L Coverage	97.32%	98.81%	89.29%	92.86%
AGL	Unconditional Cov Stat - LRuc	6.55	0.12	<b>108.79</b>	<b>54.4</b>
AGL	Independence Stat - LRind	-2.58	1.54	-36.61	-22.41
AGL	Joint Test - LRuc + LRind	3.97	1.66	<b>72.17</b>	<b>31.99</b>
PIK	% of P&L Coverage	96.73%	98.21%	98.81%	99.11%
PIK	Unconditional Cov Stat - LRuc	<b>10.99</b>	1.7	0.12	0.04
PIK	Independence Stat - LRind	3.84	1.71	1.43	1.03
PIK	Joint Test - LRuc + LRind	<b>14.83</b>	3.41	1.55	1.07
CML	% of P&L Coverage	99.09%	97.58%	99.09%	89.43%
CML	Unconditional Cov Stat - LRuc	0.03	4.81	0.03	<b>104.88</b>
CML	Independence Stat - LRind	0.17	-2.96	0.7	-19.87
CML	Joint Test - LRuc + LRind	0.2	1.84	0.73	<b>85</b>
SKY	% of P&L Coverage	99.1%	98.2%	100%	96.11%
SKY	Unconditional Cov Stat - LRuc	0.04	1.73	<b>6.71</b>	<b>16.3</b>
SKY	Independence Stat - LRind	-2.94	-3.08	<b>FAIL</b>	-11.67
SKY	Joint Test - LRuc + LRind	-2.9	-1.35	<b>FAIL</b>	4.63

Table 5.16: Statistical Backtesting Results 3

which the standard VaR models have with illiquidity and raises concerns regarding their ability to perform in times of stress when liquidation costs become more grave.

Two trends are evident for the spread-adjustment models: the Bangia model becomes less accurate as liquidity declines (failing on the basis of its 100% coverage in the case of CML and lack of independence in the case of SKY) while the François-Heude model becomes more accurate. As expected, the Angelidis model is fairly stable, only failing the conditional coverage test in the case of CML due to forecasts which, although reasonable, are always higher than the realized loss.

Sadly the promising Almgren & Chriss and Hisata & Yamai models perform badly. While they both perform well for PIK, they provide inaccurate forecasts in the case of both liquid and illiquid stocks. For the liquid stock, AGL, they fail on the basis of underestimation of risk while for the illiquids it seems that they overestimate risk. This is in keeping with the earlier contention that the models generally provide low forecasts which are prone to severe spikes that occur at the incorrect moments.

The existence of serial over and underestimation of risk points to the fact that the models' inputs are not being estimated accurately and that data "holes" are biasing their results, creating alternate runs of over-conservative and under-conservative forecasts. Given the sophistication and sensitivity of these models to their inputs, data inadequacy and poor estimation is a grave concern.

Overall, while the Al Janabi model performs the worst from all the models considered, the Shamroukh, Berkowitz, Angelidis, Monte Carlo and Parametric Mid VaR models perform fairly well. With the exception of the

Berkowitz model, they all only fail once as a consequence of poor coverage. The Berkowitz model stands out as the only model which remains accurate irrespective of the stock to which it is applied.

### 5.2.3 Synopsis

The aim of the above analysis was to assess the accuracy with which the different measures/models capture liquidity risk and to shed light on their practicability. As an addition, the model's behaviour over time was also analysed so as to understand their comparative weaknesses and drivers.

Judging the accuracy of the measures was made difficult by the fact that they are observable market variables which provide no objectively verifiable liquidity forecasts. Thus their accuracy could only be assessed by examining to what extent expected changes in their behaviour over time correlate to known periods of illiquidity.

Based on such an examination, it was shown that while the measures do not provide extremely clear trend-changes in response to changes in market-wide liquidity, there is evidence of some degree of sensitivity. The long-term graphical analysis provides tentative support for the idea that the measures do indeed capture *ex post* information regarding liquidity. There is, however, very little support for their ability as predictive liquidity metrics.

Of course, given the many different aspects which contribute to liquidity risk, it was important to determine whether the different measures provide different insights into liquidity. Although the theoretical discussion provided clear demarcations for each of the measures, these needed empirical verification.

The results of the factor analysis and the correlation analysis support the theoretical groupings of the different measures and provide evidence that different measures, do indeed capture information relating to distinct aspects of liquidity risk. While the inter-relationships between the measures and the priority which they assume in accounting for the totality of liquidity risk remains unstable across stocks, it seems undeniable that liquidity risk can only be measured by a new integrated measure or by monitoring a host of measures simultaneously.

Together the measures, however, do not offer an exhaustive or very accurate description of liquidity. Indeed they ignore many of the additional aspects of liquidity risk as noted in Section 3.6. Hopes must thus lie in a new measure

or model which provides a realistic and complete gauge of liquidity risk as represented by the transaction-cost based definition in Chapter 2.

Unfortunately the bulk of the current, integrated L-VaR models do not provide estimates of loss which are inherently more accurate than standard VaR models. Their estimates are generally prone to large and unrealistic upward spikes and exhibit volatility which is not reflected in realized portfolio losses. This brings the usefulness of their liquidity adjustments into question.

While a few of the L-VaR models like the Angelidis, Shamroukh and Berkowitz models provide more conservative estimates of loss which are stable across differing degrees of liquidity and more representative of realized portfolio losses, their estimates are just as accurate as those offered by the parametric Mid and Monte Carlo VaR models. The exception is the Berkowitz model which provides realistically stable and accurate estimates of loss even when the standard VaR models fail.

The key problem with the bulk of the L-VaR models is the excessive conservatism and undue volatility of their estimates. Both of these conditions lead them to systematically overestimate portfolio loss. This is even true of the theoretically impressive price-impact models, whose rigour and scope made them seem highly promising.

As noted in Section 3.6, a good model/measure of liquidity must offer a fair reflection of reality and should capture the bulk of the aspects which together characterize market-related liquidity risk. Unfortunately practical implementation has shown that many of the models which offer rich descriptions of liquidity provide estimates which are also the most unrealistic.

Although the reasons for each of the L-VaR model's failure are diverse, in aggregate it stems from a difficulty in estimating parameter inputs which are too important to the model's forecasts. This is true of the Bangia and the Francoise-Heude model which are too sensitive to the relative effective spread but especially true of the price impact models which, as shown by the sensitivity analysis, are highly sensitive to changes in parameters which are not easy to estimate. In this way the added complexity of the models becomes a drawback and simpler models, like Berkowitz, gain prominence.

Overall the Berkowitz model, does not specifically model the spread or price-impacts or any particular aspect of liquidity risk. All of these are subsumed in the realized change in the portfolio's value and cash flows which are used to estimate a liquidity-adjusted portfolio mean and variance.

By focusing on realized portfolio liquidity, the model bypasses much of the

problems in objectively modelling the different aspects of liquidity risk separately. While more accurate, this method does offer a very closed-box view of liquidity risk and how its different aspects affect a portfolio's risk. This could prove problematic in times of stress when portfolio trends break with realized history.

The closed-box view offered by the Berkowitz model highlights the tension in modelling between being accurate, complete and descriptive. While the Berkowitz model is accurate, it is not descriptive and while the price-impact models are descriptive and complete in that they offer insights into price-impacts, they are not accurate. The only real solution lies in using a host of measures/models simultaneously.

Based on the results presented above, it seems that the Berkowitz and/or the Angelidis L-VaR models should be used to provide a more accurate and conservative VaR forecast. In this way the major aspects of liquidity risk, the spread and the realized price impacts, are accounted for in VaR. These should be supplemented by monitoring the return-reversal measures and the Bid-Ask spread over time.

In the absence of an accurate, complete and practicable integrated measure of liquidity risk, a more complete and accurate view of liquidity can not be affected with any single measure or model.

## Chapter 6

# Conclusion

The most-recent financial recession, spurred on as it has been by market-related illiquidity with endemic effects, has undoubtedly highlighted the escalating importance of liquidity risk management in an increasingly integrated global financial system. This is evidenced by the attempts in the latest version of the Basel Accord, Basel III, to account for liquidity risk.

Although the financial literature has developed an array of measures/models which offer different, yet compelling approaches to the management of liquidity risk, such approaches have not yet gained traction in practice. This may be attributed to the fragmented approach taken by the literature in arriving at a definition of liquidity risk and a lack of awareness, on the part of market practitioners, of the importance and mechanics of liquidity risk.

The aim of this thesis is to stem some of this confusion by providing a comprehensive view of liquidity risk in terms of its many aspects, its effects and the approaches taken by the literature for its management. This is achieved by distilling a complete and robust definition of liquidity risk and by assessing a few of the liquidity-risk approaches put forward in the literature.

Unlike previous papers which have only concentrated on a theoretical discourse, the model assessment was implemented by examining the models' ability to capture the totality of the aspects which contribute to liquidity risk and by empirically testing their accuracy.

The result of the analysis shows that neither the measures nor the complex L-VaR models provide a complete and accurate view of liquidity risk.

While the measures provide useful information on the market consensus of



liquidity over time, they are not objective or exhaustive. Thus even if all the measures were used simultaneously, certain important aspects of liquidity risk like the price-risk/trade-risk trade-off, the opportunity costs associated with trade, many of the exogenous liquidity costs and the depth of price impacts, would be ignored.

The fact that empirical tests show that different measures seem to specialise in different aspects of liquidity risk complicates their use in risk management. This is compounded by the evidence that the measures are not entirely accurate in terms of their sensitivity to changes in market-wide liquidity events. Although it is clear that they do respond in line with expectations when aggregate liquidity falls, they do not seem to offer accurate, predictive information which could be used in decision-making.

The hopes presented with the development of the integrated L-VaR models was that finally a complete measure of both market and liquidity risk had been found which would provide accurate predictive information for decision-making. Unfortunately the realities are somewhat different and while many of the L-VaR models are indeed predictive and incorporate many of the aspects which contribute to liquidity risk, no single model captures all of these aspects.

Generally specific L-VaR models seem to specialise in modelling a particular aspect of liquidity risk, like the spread or price impacts, while ignoring other aspects. Thus while the models seem more comprehensive in that they have a more detailed derivation than the measures, they do not offer a complete description.

No single model covers everything and significantly no model offers adjustments which make them adaptable to times of stress when parameter estimation must change and the internal relationships between price-risk and spread-risk are modified. This is also true of the measures.

In terms of accurately forecasting portfolio losses in the face of liquidity concerns, many of the models, especially the complex ones, perform just as well as two of the standard VaR models: the parametric VaR based on the mid-price and its statistical counterpart, the Monte Carlo VaR. With the exception of the Berkowitz L-VaR, all the models perform just as well or worse than standard VaR. This casts doubts on the efficacy of their liquidity adjustments.

While many of the L-VaR models are certainly more conservative than standard VaR, they are not necessarily more representative of realized portfolio losses. This seems principally due to the heightened volatility of the L-VaR

models caused by spikes in their parameter estimates which are easily influenced by data deficiencies. This is compounded by the severe sensitivity which some of the models display to changes in their inputs.

Future models of liquidity risk must account for this unintended volatility and sensitivity in order to become more accurate. They should address the problems surrounding the estimation of parameter inputs and should account for the fact that data deficiencies with illiquid instruments could skew the models' estimates of loss. The models need to be made more robust to changes in their inputs brought about by data deficiencies.

The success which the Berkowitz model displays in statistical backtesting, highlights one of the possibilities by which future L-VaR models could become more accurate. The Berkowitz model uses realized portfolio data and no specific modelling of the different aspects of liquidity. Other models, in contrast, separate the modelling of price impacts, spread volatility, etc., incorporating them individually into their VaR estimate of loss. This leaves them more vulnerable to data deficiencies and estimation error but makes them more transparent.

In order to realize the benefits of the Berkowitz model's accuracy while retaining transparency, many of the models could be calibrated on realized portfolio data. For the OLS models, this would mean estimating the price impacts from realized trade volumes and prices relative to the market's Bid and Ask, while for the spread-adjustment models it would mean using the realized spread faced through actual trading rather than the standard relative effective spread.

Such an approach could leave the OLS models with more accurate price-impacts, while making the Bangia and François-Heude models less vulnerable to random changes in the relative effective spread brought about by a lack of Bids or Asks. Testing whether this would work is certainly an avenue for future research.

Of course many of the model's weaknesses cannot be resolved by changing the data upon which they are calibrated. Most of the L-VaR models display a marked difference in their modelling in that models which focus on the spread, tend to ignore price impacts and vice versa. Finding a model which incorporates both aspects and which does so, unlike the François-Heude model, with readily available data may prove rewarding.

Similarly investigating the estimation of price impacts and finding ways to make them more accurate would most certainly enhance the OLS models. Currently the literature offers very little guidance in terms of the estimation

of price impacts and this area is worthy of further independent research. Finding closed-form L-VaR solutions in the face of alternate price-impact formulations could also be promising.

An important question, which is not addressed in this thesis, is whether accounting for extreme-tail risk effectively accounts for liquidity-related risk. As noted the implementation of many of the Extreme Value Theory models is onerous. This, however, does not detract from the fact that comparing these models against the L-VaR models may prove insightful.

As it stands a great deal of work needs to be done in order to make the L-VaR models more representative of portfolio losses in the face of liquidity-related costs. Such work would ideally provide a more complete description of liquidity while providing greater accuracy.

Failing the existence of a truly integrated and accurate model of market-related liquidity risk, market-practitioners must persist with using an array of measures/models simultaneously. Hopefully the analysis provided in this thesis will help them in making their choice.

# Bibliography

- [1] Acharya, V. A. and Schaefer, S., (2006), *Liquidity Risk and Correlation Risk: Implications for Risk Management*, London Business School, London, Working Paper, 2006
- [2] Acharya, V. V. and Pedersen, L. H., (2000), Commonality in Liquidity, *Journal of Financial Economics*, Vol. 56, No. 1, 3-28, 2000
- [3] Acharya, V. V. and Pedersen, L. H., (2005), Asset Pricing with Liquidity Risk, *Journal of Financial Economics*, Vol. 77, No. 2, 375-410, April 2005
- [4] Aitken, M. and Comerton-Forde, C., (2003), How should liquidity be measured?, *Pacific Basin Finance Journal*, Vol. 11, No.1, 45-59, January 2003
- [5] Al Janabi, M. A. M., (2008), Integrating Liquidity Risk Factor into a Parametric Value at Risk Method, *Journal of Trading*, 76-87, Summer 2008
- [6] Almgren, R. and Chriss, N., (1999), Value Under Liquidation, *Journal of Risk*, Vol. 2, No. 4, 16-32, December 1999
- [7] Almgren, R. and Chriss, N., (2000), Optimal execution of portfolio transactions, *Journal of Risk*, Vol. 3, No. 2, 5-39, 2000
- [8] Amihud, Y., (2002), Illiquidity and stock returns: Cross-section and time-series effects, *Journal of Financial Markets*, Vol. 5, No. 1, 31-56, January 2002
- [9] Amihud, Y. and Mendelson, H., (1986), Liquidity and Stock Returns, *Financial Analysts Journal*, Vol. 42, No. 3, 43-48, June 1986
- [10] Angelidis, T. and Benos, A., (2006), Liquidity adjusted value-at-risk based on the components of the bid-ask spread, *Applied Financial Economics*, Vol. 16, No. 11, 835-851, 2006

- [11] ANON, (1996), Basel Committee on Banking Supervision, *Supervisory Framework for the Use of 'Backtesting' in Conjunction with the Internal Models Approach to Market Risk Capital Requirements* [Online], Bank for International Settlements, Available: <http://www.bis.org/publ/bcbs22.htm>, [2 February 2009]
- [12] ANON, (2008), JSE Trading Services, *JSE Trade-Elect Conceptual Training* [Online], JSE, Available: [http://www.jse.co.za/Libraries/JSE\\_-\\_ Technical\\_Library\\_-\\_ JSE\\_Trading\\_Information\\_Systems\\_-\\_ JSE\\_TradeElect\\_and\\_InfoWiz\\_Documentation\\_-\\_ Trading\\_and\\_Information\\_Specifications\\_and\\_6\\_1Guidance\\_Notes\\_-\\_ Guidance\\_Notes/J\\_TradeElect\\_Conceptual\\_Training.sflb.ashx](http://www.jse.co.za/Libraries/JSE_-_Technical_Library_-_JSE_Trading_Information_Systems_-_JSE_TradeElect_and_InfoWiz_Documentation_-_Trading_and_Information_Specifications_and_6_1Guidance_Notes_-_Guidance_Notes/J_TradeElect_Conceptual_Training.sflb.ashx), [4 January 2010]
- [13] ANON, (2010), Wikipedia The Free Encyclopaedia, *JSE Limited* [Online], Available: [http://en.wikipedia.org/wiki/JSE\\_Limited](http://en.wikipedia.org/wiki/JSE_Limited), [4 January 2010]
- [14] Bangia, A., Diebold, F., Schuermann, T. and Stroughair, J. (1999), Liquidity on the outside?, *Risk Magazine*, Vol. 12, 68-73, June 1999
- [15] Beder, T.S., (1995), VaR: Seductive but Dangerous, *Financial Analysts Journal*, Vol. 51, No. 5, 12-24, October 1995
- [16] Berkowitz, J., (2000), Breaking the Silence, *Risk Magazine*, Vol. 13, 105-108, October 2000
- [17] Berkowitz, J. and O'Brien, J., (2001), *How Accurate Are Value-at-Risk Models at Commercial Banks?*, US Federal Reserve Board, Finance & Economic Series Discussion, Working Paper, No 2001-31
- [18] Bertsimas, D. and Lo, A. W., (1998), Optimal Control of Execution Costs, *Journal of Financial Markets*, Vol. 1, No. 1, 1-50, April 1998
- [19] Black, F., (1971), Towards a fully automated exchange: Part 1, *Financial Analyst Journal*, Vol. 27, No. 1, 29-34, August 1971
- [20] Botha, M., (2008), Portfolio Liquidity-Adjusted Value-At-Risk, *South African Journal of Economics*, Vol. 11, No. 2, 203-216, 2008
- [21] Campbell, S. D., (2006), A review of backtesting and backtesting procedures, *Journal of Risk*, Vol. 9, No. 2, 1-18, 2006
- [22] Casparri, M. T., Garcia-Fronti, J. I. and Rey, S. A., (2005), *Liquidity Risk Estimation using Fuzzy Measure Theory* [Online], Finance, EconWPA, Available: <http://econpapers.repec.org/RePEc:wpa:wuwpfi:0504012>, [3 February 2009]

- [23] Çetin U., Jarrow, R.A and Protter, P., (2004), Liquidity risk and arbitrage pricing theory, *Finance and Stochastics*, Vol. 8, No. 3, 311-341, August 2004
- [24] Çetin, U., Jarrow, R.A., Protter, P. and Warachka, M., (2002), *Option pricing with liquidity risk* [Online], Available: <http://www.orie.cornell.edu/~protter/WebPapers/bspaper3.pdf>, [2 April 2009]
- [25] Chan, L.K.C. and Lakonishok, J., (1995), The Behavior of Stock Prices around Institutional Trades, *Journal of Finance*, Vol. 50, No. 4, 1147-1174, September 1995
- [26] Chen, J., (2005), *Pervasive Liquidity Risk And Asset Pricing*, Graduate School of Business, Columbia University, New York, Working Paper, 2005
- [27] Chincarini, L., (2008), A Case Study on Risk Management: Lessons from the Collapse of Amaranth Advisors L.L.C., *Journal of Applied Finance*, Vol. 18, No. 1, 2008
- [28] Chollete, L., Næs, R. and Skjeltorp, J. A., (2007), *What captures liquidity risk? Order based versus trade based liquidity measures*, The VGSF & NHH Skinance Workshop, Bergen Norway, 19-21 March 2007
- [29] Chollete, L., Næs, R. and Skjeltorp, J. A., (2008), *The Risk Components of Liquidity*, Norges Bank, Oslo, Working Paper, 2008/03, March 2008
- [30] Chordia T., Roll, R. and Subrahmanyam A., (2000), Order Imbalance, Liquidity and Market Returns, *Journal of Financial Economics*, Vol. 65, 111-130, 2002
- [31] Chordia T., Shivakumar L. and Subrahmanyam A., (2004), Liquidity Dynamics Across Small and Large Firms, *Economic Notes*, Vol. 33, No. 1, 111-143, February 2004
- [32] Danielsson, J. and De Vries, C.G., (2000), Value-at-Risk and Extreme Returns, *Annales d'Économie et de Statistique*, No. 60, Financial Market Microstructure, 239-270, December 2000
- [33] Danielson, J. and Zigrand, J. P., (2006), On time-scaling of risk and the square-root-of time rule, *Journal of Banking & Finance*, Vol. 30, No. 10, 2701-2713, October 2006
- [34] De Raaji, G. and Raunig, B., (1998), A Comparison of Value at Risk Approaches and Their Implications for Regulators, *Focus on Austria*, Vol. 4, 57-71, December 1998

- [35] Demsetz, H., (1968), The Cost of Transacting, *Quarterly Journal of Economics*, Vol. 82, No. 1, 33-53, February 1968
- [36] Diebold, F.X. and Lopez, J., (1995), Modelling volatility dynamics, (In Horner, K. (ed.), *Macroeconomics: Developments, Tensions and Prospects*, Boston MA: Kluwer Academic Press, 427-472)
- [37] Domowitz, I., Glen, J. and Madhavan, A., (2001), Liquidity, Volatility, and Equity Trading Costs Across Countries and Over Time, *Journal of International Finance*, Vol. 4, 221-255, 2001
- [38] Dowd, K., (2008), Backtesting Market Risk Models, (In Fabozzi, F. J. (ed.), *Handbook of Finance*, New Jersey: Wiley)
- [39] Dubil, R. (2003), How to Include Liquidity in a Market VaR Statistic?, *Journal of Applied Finance*, Vol. 13, No. 1, 19-28, 2003
- [40] Duffie, D. and Ziegler, A., (2001), Liquidation Risk, *Financial Analysts Journal*, Vol. 59, No. 3, 42-51, May 2003
- [41] Eckbo, E. B. and Norli, Ø., (2002), *Pervasive Liquidity Risk*, Tuck School of Business, Dartmouth College, New Hampshire, Working Paper, November 2002
- [42] Fernandez, Frank A. (1999), *Liquidity Risk New Approaches to Measurement and Monitoring*, Securities Industry Association, New Jersey, Working Paper, 1999
- [43] Fleming, J.M., (2003), *Measuring Treasury Market Liquidity*, Federal Reserve Bank of New York, Economic Policy Review, Vol. 9, No. 3, Working Paper, September 2003
- [44] François-Heude, A. and Van Wynendale, P., (2002), *Integrating liquidity risk in a parametric intraday VaR framework*, 7th Bergian Financial Research Forum, Liège
- [45] Giesecke, K. and Goldberg, L.R., (2005), Forecasting Extreme Financial Risk, (In Ong, M. (ed.), *Risk Management: A Modern Perspective*, San Diego: Wiley, 609-536)
- [46] Glosten, L. and Harris, L.E., (1988), Estimating the Components of the Bid-Ask Spread, *Journal of Financial Economics*, Vol. 21, No. 1, 123-142, May 1988
- [47] Goodhart, C. (2008), Liquidity risk management, *Banque de France Financial Stability Review – Special issue on liquidity*, Vol. 11, 39-44, February 2008

- [48] Grossman, S. J. and Miller, M. H., (1988), Liquidity and Market Structure, *Journal of Finance*, Vol. 43, No. 3, 617-633, July 1988
- [49] Hisata, Y. and Yamai, Y., (2000), *Research Toward the Practical Application of Liquidity Risk Evaluation Methods*, Institute for Monetary and Economic Studies, Bank of Japan, Discussion Paper, Vol. 18, No. 2, 83-127, December 2000
- [50] Holthausen, R. W., Leftwich, R. W and Mayers, D., (1987), The Effect of Large Block Transactions on Security Prices: A Cross-Sectional Analysis, *Journal of Financial Economics*, Vol. 19, 237-268, 1987
- [51] Hu, S., (1997), *Trading Turnover and Expected Stock Returns: The Trading Frequency Hypothesis and Evidence from the Tokyo Stock Exchange*, National Taiwan University and University of Chicago, Working Paper, 1997
- [52] Huang, A. Y., (2009), A Value at Risk Approach with Kernel Estimator, *Applied Financial Economics*, Vol. 19, No.5, 379-395, 2009
- [53] Hull, J.C, (2006), *Options, Futures, and Other Derivatives*, 6th rev. ed., New Jersey: Prentice Hall
- [54] Hurlin, C. and Tokpavi, S., (2006), Backtesting VaR Accuracy: A Simple New Test, *Journal of Risk*, Vol. 9, No. 2, 19-39, April 2006
- [55] Jarrow, R.A. and Subramanian, A., (2001), The Liquidity Discount, *Mathematical Finance*, Vol. 11, No. 4, 447-474, October 2001
- [56] Jorion, P., (2000), Value at Risk: The New Benchmark for Managing Financial Risk, 3rd rev. ed., New York: McGraw Hill
- [57] Kamara, A. (1994), Liquidity, Taxes, and Short-Term Treasury Yields, *Journal of Financial and Quantitative Analysis*, Vol. 29, No. 3, 403-417, September 1994
- [58] Keene, M. A., Peterson, D. R., (2007), The Importance of Liquidity as a Factor in Asset Pricing, *The Journal of Financial Research*, Vol. XXX, No. 1, 91-109, Spring 2007
- [59] Kyle, A., (1985), Continuous auctions and insider trading, *Econometrica*, Vol. 53, No. 6, 1315-1335, November 1985
- [60] Le Saout, E., (2002), *Incorporating Liquidity Risk in Value at Risk Models*, Rennes University, Working Paper, 2002
- [61] Lee, K., (2005), *The World Price of Liquidity Risk*, Department of Finance, Fisher College of Business, Ohio State University, Ohio, Working Paper, August 2005



- [62] Lee, C. and Ready, M., (1991), Inferring Trade Direction from Intraday Data, *Journal of Finance*, Vol. 46, No. 2, 733-46, June 1991
- [63] Lee, M. C., Swaminathan, B., (2000), Price momentum and trading volume, *Journal of Finance*, Vol. 55, No. 5, 2017-2069, October 2000
- [64] Lo, A., Petrov, C. and Wierzbicki, M., (2003), It's 11pm – Do You Know Where Your Liquidity Is? The Mean-Variance-Liquidity Frontier, *Journal of Investment Management*, Vol. 1, 55-93, 2003
- [65] Loebnitz, K., 2006, *Market Liquidity Risk Elusive: No More Defining and quantifying market liquidity risk*, Utrecht, The Netherlands, University of Twente, MSc
- [66] Lybek, T. and Sarr, A., (2002), *Measuring Liquidity in financial Markets*, IMF Working Paper, Working Paper, 02/232, 2002
- [67] McNeil, A. J., Frey, R. and Embrechts, P., (2005), *Quantitative Risk Management: Concepts, Techniques, Tools Princeton Series in Finance*, New Jersey: Princeton University Press
- [68] Morris, S. and Shin, H.S., (2004), Liquidity Black Holes, *Review of Finance*, Vol. 8, No. 1, 1-18, 2004
- [69] Muranaga, J., and Ohsawa, M. (1998), *Measurement of liquidity risk in the context of market risk calculation*, mimeo, Institute for Monetary and Economic Studies, Bank of Japan.
- [70] Pástor, L. and Stambaugh, R. F., (2003), Liquidity and Expected Stock Returns, *The Journal of Political Economy*, Vol. 111, No. 3, 642-685, June 2003
- [71] Patel, B., (2001), Measurement of Transaction Costs in Equity Trading, *Transaction Costs, Institutional Investor Investment Guides*, 49-54, Spring in Loebnitz, K. (2006)
- [72] Porter, R. B., (2003), *Measuring Market Liquidity*, University of Florida, Florida, Working Paper, October 2003
- [73] Refenes, A. P. N., Skiadopoulos, G. and Skintzi, V. D., (2005), *Journal of Alternative Investments*, Vol. 7, No. 4, 66-82, 2005
- [74] Rogachev, A., (2002), The Value-at-risk concept by Swiss private banks, *The Journal of Risk Finance*, Vol. 8, No. 1, 72-78, 2007
- [75] Shah, A. and Thomas, S., (2007), *Measuring Liquidity* [Online], Available: [http://www.cse.lk/270808/pdf/6\\_measuring\\_liquidity.pdf](http://www.cse.lk/270808/pdf/6_measuring_liquidity.pdf), [3 March 2009]

- [76] Shamroukh, N., (2000), *Modelling liquidity risk in VaR models*, Algoritmics, UK, Working Paper, 2000
- [77] Stigler, G.J., (1964), Public Regulation of Securities Market, *Journal of Business*, Vol. 37, No. 2, 117-142, April 1964
- [78] Ting, C. and Warachka, M., (2003), *A New Methodology for Measuring Liquidity-induced Transaction Costs*, School of Business Working Paper, Singapore Management University, Singapore, 2003
- [79] Upper, C., (2000), *Measuring Liquidity Under Stress*, Bank for International Settlements, Working Paper, No. 2, 2000

## Appendix A

# The Average-Weighted Spread

The AWS at time  $t$  depends on the average-weighted bid and ask prices at time  $t$ . These represent the expected cost/gain from trading an amount equal to the NMS at time  $t$ .

If the NMS of an asset is 15 000 securities and at time  $t$  the order book looks as below in Figure A.1 then the AWS is found by calculating the weighted-average bid/ask price, assuming that 15 000 units must be bought/sold. For the order book below, this translates into a weighted-average bid of 627 and ask of 631, implying a AWS of  $631 - 627$ .

CAC Central Market			
XYZ		10.49 a.m.	
TNB 15000		WAS 627 - 631	
Buy-orders		Sell orders	
Quantity	Price	Price	Quantity
3200	629	630	2560
2330	628	631	1780
2270	627	632	5020
2530	626	633	6370
4760	625	634	5870
Bid : $\frac{(3200*629) + (2330*628) + (2270*627) + (2530*626) + (4670*625)}{15000} = 627$			
Ask : $\frac{(2560*630) + (1780*631) + (5020*632) + (5640*633)}{15000} = 631$			

Figure A.1: Weighted-Average Spread[60].

## Appendix B

# GMM Estimation

The Angelidis *et al* [10] model requires estimation of  $(\theta, \phi, \kappa)$  using only intra-day trade prices  $P_t$ , associated volumes  $V_t$  and trade directions  $X_t$ . They propose that GMM estimation be used to find these estimates as it requires weaker distributional assumptions than maximum likelihood methods or ordinary least squares (OLS) regression. While OLS regression remains valid, they prefer GMM estimation.

GMM is based loosely on the sample method of moments estimation procedure.

### The Sample Method of Moments

In general the  $r$ th algebraic moment of a random variable  $X$  with distribution function  $X$  is

$$\mu_r = \int_X x^r f(x) dx \quad (\text{B.1})$$

Now a sensible estimator of  $\mu'_r$  from a sample  $\{x_i\}_{i \in \mathbb{N}}$  of realizations of  $X$  is

$$\mu_r = \frac{1}{n} \sum_{i=1}^n x_i^r \quad (\text{B.2})$$

This can be used to estimate any theoretical relationship between a parameter of a distribution by evaluating  $\mu'_r$  with the sample and then solving for

the parameter  $a$  by using:

$$\bar{\mu}'_r = M(a) \quad (\text{B.3})$$

where  $M$  is the theoretical relationship between  $a$  and  $\mu'_r$ , the true moment.

### Generalized Method of Moments

GMM generalizes the above simple method by assuming that based on theory or specification a function  $f(X_t, \theta)$  exists such that  $\mathbb{E}[f(X_t, \theta)] = 0$  for some unknown parameter value  $\theta$ .

GMM assumes that the best estimate of  $\theta$  is the one that sets this expectation to zero. Here  $f$  is a known function of the random variable  $X$  and  $\theta$  is a parameter of the distribution of  $X$ . The goal of GMM is to find  $f$  based on theory such that  $\mathbb{E}[f(X_t, \theta)] = 0$ .

The GMM estimator of  $\theta$ , the parameter vector, for realizations  $X_i$  and some positive definite weighting matrix  $C$ , is found by minimizing:

$$\hat{\theta} = \min_{\theta} \frac{1}{N} \left[ \sum_{i=1}^N f(X_i, \theta) \right]^t C \left[ \sum_{i=1}^N f(X_i, \theta) \right] \quad (\text{B.4})$$

Generally  $C$  takes the form of an inverse matrix based on some initial estimate  $\hat{\theta}_{int}$  of  $\theta$

$$C = \frac{1}{N} \left( \sum_{i=1}^N f(X_i, \hat{\theta}_{int}) \cdot f(X_i, \hat{\theta}_{int})^t \right)^{-1} \quad (\text{B.5})$$

Now in the Angelidis *et al* [10] model the estimation equation is based on relating price changes  $p_t - p_{t-1}$  to volumes, trade directions: and white noise  $u_t$ :

$$p_t - p_{t-1} = \theta \sqrt{V_t} (X_t - \rho X_{t-1}) + \phi (X_t - X_{t-1}) + \kappa (X_t \sqrt{V_t} - X_{t-1} \sqrt{V_{t-1}}) + u_t \quad (\text{B.6})$$

Angelidis *et al* [10] propose the use of the following expectation equation to implement GMM:

$$\mathbb{E} \begin{pmatrix} X_t X_t - \rho X_{t-1}^2 \\ u_t - \alpha \\ (u_t - \alpha) X_t \sqrt{V_t} \\ (u_t - \alpha) X_{t-1} \sqrt{V_{t-1}} \\ (u_t - \alpha) \sqrt{V_t} \\ (u_t - \alpha) \sqrt{V_{t-1}} \end{pmatrix} = 0$$

These equations are based on the standard OLS regression assumptions with constant  $\alpha$ .

University of Cape Town

## Appendix C

# Statistical Backtesting of Value-at-Risk

The Basel accord allows banks and other regulated entities to develop their own internal models for the management of risk. Thus while the accord specifically recommends the use of Value-at-Risk models and provides special attention to the statistical backtesting of VaR in its Supervisory Framework [11], users have tended to follow their own specifications in backtesting their risk models. This has had the effect of creating a plethora of, at times, inconsistent methodologies for the backtesting of VaR.

Despite this array of methodologies, the literature agrees that backtesting of Value-at-Risk should, at the very least, amount to testing two hypothesis regarding the “hit rate”<sup>1</sup> of a VaR model [54]:

1. The unconditional coverage hypothesis which points out that a  $\lambda\%$  VaR should be breached close to  $1 - \lambda\%$  of the time by realized portfolio losses. The VaR model should thus not systematically under or overestimate risk.
2. The independence hypothesis which argues that violations of the VaR forecast should over time be independent and not autocorrelated. Thus the occurrence or non-occurrence of a violation should bear no information on future violations. If this is true then the VaR model should not have difficulty in forecasting changes in the profit and loss distribution.

---

<sup>1</sup>The hit rate of a VaR model represents the number of times that the VaR forecast underestimates true loss

The above criteria can be expressed mathematically. Let  $R_t$  be the return on a portfolio at time  $t$ . Then the VaR of the portfolio using a  $\lambda\%$  coverage level at time  $t$ ,  $VaR_{t|t-1}(\lambda)$ , and adapted to an information set  $\Omega_{t-1}$  is such that

$$\mathbb{P}(R_t < VaR_{t|t-1}(\lambda)) = \lambda \quad (C.1)$$

Let now  $I_t(\lambda)$  be an *ex-post* indicator variable such that

$$I_t(\lambda) = \begin{cases} 1 & \text{if } R_t < VaR_{t|t-1}(\lambda) \\ 0 & \text{else} \end{cases}$$

then  $(I_t(\lambda))_{t \in T}$  is called the hit series associated to the VaR and defines whether or not the realized loss is larger than that forecasted by the VaR model [54].

In this setting the problem of evaluating  $VaR_{t|t-1}(\lambda)$  in terms of the initial criteria reduces to the task of determining if the hit series has the following properties:

1. The conditional coverage hypothesis –  $\mathbb{P}(I_t = 1) = \mathbb{E}(I_t(\lambda)) = 1 - \lambda$  so the probability of an *ex-post* violation must exactly equal the model's coverage rate.
2. The independence hypothesis –  $I_t(\lambda)$  must be independent of  $I_{t-k}(\lambda)$  for any  $k \neq 0$ .

While a number of tests, using a range of insights, have been developed to make the testing of these hypothesis tractable, the most widely used tests are the Kupiec Binomial test for conditional coverage and the Christoffersen joint independence test [21].

The Kupiec test is based on the idea that if the hit series represents the realization of independent identically distributed random variables and if the VaR model is accurate then the number of violations should follow a binomial distribution with probability  $\lambda$ . The probability then of arriving at  $\alpha$  violations given  $n$  observations is given by  $Prob(\alpha | n, \lambda) = \binom{n}{\alpha} \lambda^\alpha (1 - \lambda)^{n-\alpha}$ .

The above can be used to accept or reject the null hypothesis of model accuracy by calculating the probability of achieving more than some empirical level of violations using the binomial formula with  $n$  and  $\lambda$  [38].



The Kupiec test, however, assumes independence as a precondition for the test. The breakthrough which Christoffersen had was to realized that the two hypothesis can be tested for separately and jointly. Arguing that the two conditions imply that  $(I(\lambda))_{t \in T}$  is i.i.d Binomial with mean of  $\lambda$ , he shows that if  $(I(\lambda))_{t \in T}$  is a first-order Markov process then its one-step ahead transition probabilities  $\mathbb{P}(I_t | I_{t-1})$  can be defined as:

$$\mathbf{\Pi} = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}$$

where  $\pi_{ij} = \mathbb{P}(I_t(\lambda) = j | I_{t-1}(\lambda) = i)$ .

Modelling  $(I(\lambda))_{t \in T}$  in this way amounts to assuming that a violation one time-step ago bears no information on a violation in the next time step. The test then tests for the existence of “an order one memory process” [54].

The test is actually a joint test of independence and conditional coverage with  $H_0$  taking the form:

$$H_0 : \mathbf{\Pi} = \mathbf{\Pi}_\lambda = \begin{pmatrix} 1 - \lambda & \lambda \\ 1 - \lambda & \lambda \end{pmatrix}$$

The test statistic of this test, like the test itself, can be split into two parts:  $LR_{UC}$  and  $LR_{IND}$  both of which are distributed  $\chi^2(1)$  and have the following form:

$$LR_{UC} = -2 \ln[(1 - \lambda)^{n-\alpha} \lambda^\alpha] + 2 \ln[(1 - \frac{\alpha}{n})^{n-\alpha} (\frac{\alpha}{n})^\alpha] \quad (C.2)$$

$$LR_{IND} = -2 \ln[(1 - \hat{\pi}_2)^{n_{00}+n_{11}} \hat{\pi}_2^{n_{01}+n_{11}}] + 2 \ln[(1 - \hat{\pi}_{01})^{n_{00}} \hat{\pi}_{01}^{n_{01}} (1 - \hat{\pi}_{11})^{n_{10}} \hat{\pi}_{11}^{n_{11}}] \quad (C.3)$$

where  $\pi_{01} = \frac{n_{01}}{n_{00}+n_{01}}$ ,  $\pi_{11} = \frac{n_{11}}{n_{10}+n_{11}}$  and  $\hat{\pi}_2 = \frac{n_{01}+n_{11}}{n_{00}+n_{10}+n_{01}+n_{11}}$  are estimated using  $n_{ij}$  from the empirical hit series itself.  $n_{ij}$  here represents the number of times that event  $j$  has occurred consecutively after the observance of event  $i$  in the sample. Generally calculating this value for a long series is not easy and is best done with code. The VBA code for finding these values is presented in Appendix D.

Now since both  $LR_{IND} \sim \chi^2(1)$  and  $LR_{UC} \sim \chi^2(1)$  then the joint test  $LR_{CC} = LR_{IND} + LR_{UC}$  is distributed  $\chi^2(2)$  [54].

Overall the joint test is easy to determine and, although it only tests for serial dependencies of order 1, is widely used by most VaR practitioners in determining the VaR accuracy for capital adequacy purposes [38].

University of Cape Town

## Appendix D

### VBA Code

The VBA Sub *Monte Carlo* implements a simulation of 10 000 returns based on a Geometric Brownian motion for the calculation of the standard Monte Carlo VaR discussed in Section 5.1.3. The code can be easily applied in a different VBA module by changing the sheet references from which input means and variances are drawn.

The VBA Function *Joint(ProL, VARn, k, m)* is used to calculate the number  $km$  states in a VaR model's hit series as discussed in Appendix C. The function reads in a range of realized Profits & Losses and a range of VaR estimates of loss and then calculates the number of incidences of state  $m$  after state  $k$  is observed. It references the function *StateCount* which actually counts the number of  $km$  states.

```

Sub Monte_Carlo()

Dim T As Integer           'Number of VAR estimates needed as outputs
Dim N As Integer           'Number of iterations of MC simulation

N = 10000
Dim R(1 To 10000) As Double 'Vector of simulated returns
T = 368

Dim Mu(1 To 368) As Double  'Vector of rolling return averages
Dim std(1 To 368) As Double 'Vector of rolling standard deviations

Dim eps As Double

Dim Percentile(1 To 368) As Double

Dim P(1 To 368) As Double    'Vector of portfolio values
Dim VAR(1 To 368) As Double  'Vector of Value-at-Risks

'Populating mean and stdev vectors
For i = 1 To T
    Mu(i) = Sheets("sky").Range("ck33:ck33").cells(i, 1)
    std(i) = Sheets("sky").Range("cl33:cl33").cells(i, 1)
    P(i) = Sheets("sky").Range("z33:z33").cells(i, 1)
Next i

For i = 1 To T
    'Populating vector of simulated returns
    For j = 1 To N
        Application.Volatile
        rand = Rnd 'Random uniform number
        eps = Application.WorksheetFunction.NormSInv(rand) 'Random Normal dbn
        R(j) = (Mu(i) - std(i) ^ 2 * 1 / 2) + std(i) * eps 'Return simulation
    Next j

    Percentile(i) = Application.WorksheetFunction.Percentile(R, 1 - 0.99)
    VAR(i) = P(i) * (Exp(Percentile(i)) - 1)
Next i

For i = 1 To T
    Sheets("sky").Range("Ar33:Ar33").cells(i, 1).Value = VAR(i)
Next i

End Sub

```

```

Function Joint(ProL As Range, VARn As Range, k As Integer, m As Integer)
'Function reads in a range of Profit/Losses _
and the VaR estimate and then determines the number of km states.

Dim PL() As Variant
Dim VAR() As Variant
Dim Indc() As Integer

Dim cell As Object
Dim l1 As Long
Dim n1 As Long

Dim counter As Integer
Dim nCount As Integer
Dim i As Integer

'Populating PL Array
counter = 1
For Each cell In ProL
ReDim Preserve PL(1 To counter)
PL(counter) = cell.Value
counter = counter + 1
Next cell

'Populating VaR Array
counter = 1
For Each cell In VARn
ReDim Preserve VAR(1 To counter)
VAR(counter) = cell.Value
counter = counter + 1
Next cell

l1 = UBound(PL)
n1 = LBound(PL)

ReDim Indc(1 To l1)

'Populating hit series Ind
For i = n1 To l1
If PL(i) > 0 Then Indc(i) = 0
If PL(i) < 0 Then If PL(i) < VAR(i) Then Indc(i) = 1 Else Indc(i) = 0
Next i

Joint = StateCount(Indc, k, m)

End Function

```

```

Function StateCount(Ind() As Integer, k As Integer, m As Integer)

Dim cell As Object
Dim l1 As Long
Dim n1 As Long

Dim counter As Integer
Dim nCount As Integer
Dim lCount As Integer
Dim i As Integer

Dim sumI As Integer
Dim FirstCons As Integer

l1 = UBound(Ind)
n1 = LBound(Ind)

'Counting states

'Handling 00 and 11 states
If k = m Then
nCount = 0
For i = n1 + 1 To l1
    If Ind(i - 1) = k Then If Ind(i) = m Then nCount = nCount + 1
Next i
End If

'Handling 01 and 10 states
'Counting sum of 1s
sumI = 0
For i = n1 To l1
    sumI = Ind(i) + sumI
Next i

'Determining FirstCons, number of 1st consecutive entries in array
lCount = 0
i = n1
Do Until Ind(i) <> m
    lCount = lCount + 1
    i = i + 1
    If lCount = 11 Then Exit Do
Loop

If Ind(n1) <> m Then FirstCons = 0
If Ind(n1) = m Then FirstCons = lCount

'Determining the function output
If k = m Then StateCount = nCount
If k <> m Then If m = 1 Then StateCount = sumI - FirstCons
If k <> m Then If m = 0 Then StateCount = 11 - sumI - FirstCons

End Function

```